Abelian varieties over ample fields (joint work with Arno Fehm)

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We say a field K **has property (AIR)** if $\mathrm{rk}(A(K)) = \infty$ for every non-zero abelian variety A/K.

Theorem 0.1. *Here are a few examples of (AIR) fields. The (AIR) property is proved with various different methods.*

- *(Mattuck): Every finite extension $E/\mathbb{Q}_p$ has property (AIR)*
- *well-known: Every alg. closed field has property (AIR) except for the $\overline{\mathbb{F}_p}$.*
- *(Frey-Jarden): Let K be a finitely generated infinite field: For almost all $\sigma \in \mathrm{Gal}(K)$ the field $\overline{K}(\sigma)$ has property (AIR).*

Conjecture 0.2. (Larsen) *Let K be a field which is not algebraic over a finite field. If $\mathrm{Gal}(K)$ is a finitely generated profinite group, then K is an (AIR) field.*

Conjecture 0.3. (Frey) *For every number field K the field K_{ab} is (AIR).*

Definition 0.4. *A field K is said to be* **ample** *if for every smooth curve C/K we have $C(K) = \emptyset$ or $|C(K)| = \infty$.*

The above (AIR) fields (or conjectured (AIR) fields) are all ample (or conjectured to be ample)!

Theorem 0.5. (Fehm-S.P.) *Every ample field K which is not algebraic over a finite field has the property (AIR).*

We published this with the additional assumption that $\mathrm{char}(K) = 0$ in 2010. We had an "almost proof" of the full result already in 2011, but this "almost proof" used a paper that turned out to contain gaps (and even a false statement). We could not close the gaps, so we did not submit the "almost proof", of course. In

November 2019 Moshe Jarden brought a new paper of Damian Roessler to our attention that contains a valid proof for the results we need. With the help of that we could finish up the proof of the above theorem.

Example 0.6. The theorem gives new examples of (AIR) fields.

- For every Henselian local domain R the quotient field $K = \mathrm{Quot}(R)$ is ample (due to Pop), hence (AIR).
- For number field K and every finite set S of places and almost all $\sigma \in \mathrm{Gal}(K)$ the field $K_{S,tot} \cap \overline{K}(\sigma)$ is ample (due to a result of Geyer-Jarden) and hence (AIR).

The theorem relates a conjecture of Koenigsmann with the above conjecture of Larsen.

Conjecture 0.7. (Koenigsmann) *Let K be a field. If $\mathrm{Gal}(K)$ is finitely generated, then K is ample.*

Corollary 0.8. *Koenigsmann's conjecture implies Larsen's conjecture.*

1 The Mordell-Lang conjecture in dimension one

Theorem 1.1. (Mordell-Lang in dimension one) *Let $K/\overline{\mathbb{F}}_p$ be an algebraically closed field and A/K an abelian variety. Let C be a subcurve of A and let $\Gamma \subset A(K)$ be a subgroup of finite rank. If $|C(K) \cap \Gamma| = \infty$, then $g_C = 1$ or C is $K/\overline{\mathbb{F}}$-isotrivial (i.e. birational to a curve over $\overline{\mathbb{F}}_p$).*

This is a special case of the following conjecture.

Conjecture 1.2. (Mordell-Lang conjecture) *Let X be a subvariety of A. If $|C(K) \cap \Gamma| = \infty$, then X is special. (A curve C on A is known to be special iff it is isotrivial or $g_C = 1$.)*

The proof of the proposition combines two important results: Ghioca-Moosa-Scanlon reduction and a recent finiteness theorem of Roessler for curves.

Theorem 1.3. (Ghioca-Moosa-Scanlon reduction) *A and X are defined over a finitely generated subfield $F/\mathbb{F}_p$ of K and in the proof of the Mordell-Lang conjecture one may assume $\Gamma \subset A(F_{\mathrm{ins}})$. Thus, in the proof of the Theorem 1.1 one can assume $|C(F_{\mathrm{ins}})| = \infty$.*

The other important ingredient is the following result of Roessler.

Theorem 1.4. (Roessler, 2019) *Let F/F_0 be a finitely generated extension of fields. Let C/F be a curve of genus ≥ 2. If $|C(F_{\mathrm{ins}})| = \infty$, then $C_{\overline{F}}$ is $\overline{F}/\overline{F}_0$-isotrivial.*

In fact Roessler proved this in "only" in the case where F_0 is algebraically closed, $\boxed{\mathrm{trdeg}(F/F_0) = 1}$ and C is smooth projective. The latter assumption is not necessary because C becomes birational to a smooth projective curve over a finite inseparable extension of F. So assume C smooth and projective.

A marvellous argument of Arno shows that the other assumptions are not necessary, too: Let $\mathscr{F}$ be the set of all intermediate fields F' of $\overline{F}/\overline{F}_0$ with $\mathrm{trdeg}(F/F') = 1$. Roessler's theorem implies that $C_{\overline{F}}$ is $\overline{F}/F'$-isotrivial for every $F' \in \mathscr{F}$. Hence, if $c \in M_g(\overline{F})$ is the point in the coarse moduli space corresponding to C, then in fact

$$c \in \bigcap_{F' \in \mathscr{F}} M_g(F') = M_g(\bigcap_{F' \in \mathscr{F}} F') = M_g(\overline{F}_0).$$

Hence $C_{\overline{F}}$ is $\overline{F}/\overline{F}_0$-isotrivial.

2 Proof of the main theorem

I shall now explain a proof of the following theorem, based on Theorem 1.1.

Theorem 2.1. *Let F be a field which is not algebraic over a finite field. Let A/F be an abelian variety. There exists a smooth curve C_A/F of genus ≥ 2 such that $C_A(F) \neq \emptyset$ and such that for every extension F'/F the implication*

$$|C_A(F')| = \infty \Rightarrow \mathrm{rk}(A(F')) = \infty$$

holds true. In particular, if F is ample, then $\mathrm{rk}(A(F)) = \infty$.

Replacing A by $A \times A$ we can assume $\boxed{\dim(A) \geq 2}$. By Theorem 1.1 it is enough to construct a curve C of genus ≥ 2 on A such that $C(F)$ contains a smooth point and such that $C_{\overline{F}}$ is not $\overline{F}/\overline{\mathbb{F}}_p$-isotrivial.

Lemma 2.2. *Let K/K_0 be an extension of algebraically closed fields. Let $f : C \to D$ be a finite morphism of smooth projective F-curves. If C is K/K_0-isotrivial, then D is K/K_0-isotrivial.*

Proof. We know that there is a sm. proj. curve C_0/K_0 such that $C \cong C_{0,K}$. Let $J_0 = J_{C_0}$. Case $g_D = 0$: Obvious.

Case $g_D = 1$: Then D is an elliptic curve which is isogenous to an abelian subvariety B of $J_{0,K}$. Thus there exists an abelian subvariety B_0 of J_0 with $B = B_{0,K}$ (cf. Conrad 2006, Milne's article). So there is an isogeny $B_{0,K} \to D$. The kernel of this isogeny is defined over K_0 as well! (cf. Conrad 2006; subtle fact using $\dim(B) = 1$). Thus the isogeny induces an isomorphism of D with a curve defined over K_0.

Case $g_D \geq 2$: There exists an intermediate field L of K/K_0, finitely generated over K_0, such that f, C, D are defined over L. Then $C(L) = C_0(L) \supset C_0(K_0)$ is infinite, because K_0 is alg. closed. As $f : C(L) \to D(L)$ has finite fibres, $D(L)$ is infinite as well. Roessler's theorem (or Grauert-Manin) imply that D is isotrivial.

Lemma 2.3. *Let F/F_0 be a transcendental extension of fields. Then there exists a smooth projective curve D/F with $g_D \geq 2$, $D(F) \neq \emptyset$ and such that $D_{\overline{F}}$ is not $\overline{F}/\overline{F}_0$-isotrivial*

Sketch of Proof. Let $j \in F$ be an element which is transcendental over F_0. One can write down an elliptic curve E with j-invariant j. This curve E is then not isotrivial. Now write down $D \to E$ a sufficiently ramified cover that is purely ramified at a F-rational pt. of E. Then D has an F-rational point, has genus ≥ 2 by Hurwitz formula and is non-isotrivial by the previous Lemma. $\square$

Lemma 2.4. *Let Y/F be a variety of dimension m and $y \in Y(F)$ a smooth point. Then there exists an affine open neighbourhood Y' of y and an immersion $f : Y' \to \mathbb{A}^{m+1}$ with $f(y) = 0$.*

Proof. By the smoothness there is an open affine neighbourhood and a étale morphism $f : Y' \to \mathbb{A}^m$ sending y to 0. This is locally standard étale. Thus there exists an open affine neighbourhood $U = \mathrm{Spec}(R)$ of 0 and after shrinking Y' we have $Y' = \mathrm{Spec}(R[T]/h(T))$ and f is isomorphic to the canonical map. Y' is obviously a subscheme of $\mathbb{A}^{m+1}$. $\square$

Lemma 2.5. *Let F/F_0 be transcendental. Let X be an F-variety with $n := \dim(X) \geq 2$ and let $x \in X(F)$ be a smooth point of X. Then there exists a closed subcurve C of X through x such that $g_C \geq 2$ and such that $C_{\overline{F}}$ is not $\overline{F}/\overline{F}_0$-isotrivial and such that C is smooth in x.*

Proof. By Lemma 2.4 there is an open subscheme D' of the curve D from Lemma 2.3 such that there exists a point $d \in D'(F)$ and an immersion $D' \to \mathbb{A}^2 \to \mathbb{A}^n$ sending d to 0. Moreover there is an open affine neighbourhood X' of x in X and an étale $X' \to \mathbb{A}^n$ sending $x \to 0$. Let C' be the connected component of $D' \times_{\mathbb{A}^n} X'$; its Zariski closure C in X will do the job. $\square$

Applying the last lemma with $F_0 = \overline{\mathbb{F}}_p$, $X = A$, $x = 0$ we get the missing piece of information for the proof of the above Theorem 2.1.