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LOCAL TO GLOBAL PRINCIPLES FOR HOMOMORPHISMS OF ABELIAN SCHEMES

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Dedicated to Moshe Jarden with admiration on the occasion of his 80th birthday

ABSTRACT

Let A and B be abelian varieties defined over the function field $k(S)$ of a smooth algebraic variety S/k. We establish criteria, in terms of restriction maps to subvarieties of S, for existence of various important classes of $k(S)$-homomorphisms from A to B, e.g., for existence of $k(S)$-isogenies. Our main tools consist of Hilbertianity methods, Tate conjecture as proven by Tate, Zarhin and Faltings, and of the minuscule weights conjecture of Zarhin in the case when the base field is finite.

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1. Introduction

Let S be a smooth variety over a finitely generated field k of arbitrary characteristic. Let $\mathscr{A}$ and $\mathscr{B}$ be S-abelian schemes with generic fibers A and B (respectively) defined over the function field $k(S)$. In this paper we consider existence of certain classes of $k(S)$-homomorphisms from A to B, e.g., $k(S)$-isogenies, and provide local criteria in terms of restriction maps to subvarieties of S. Furthermore, we study existence of abelian subvarieties of A in a similar way. Our first main result is the following local to global principle.

Theorem A (Thm. 4.5): *Let S be a smooth variety over a finitely generated field k. Let $\mathscr{A}, \mathscr{B}$ be abelian schemes over S with generic fibers A and B, respectively. Let U be a dense open subscheme of S. Let $m \in \{0, 1, \ldots, \dim(S)\}$. Assume that k is infinite or that $m \geq 1$. Let $\kappa \in \mathbb{N}$.*

(a) *The following are equivalent:*
   (i) *There exists a $k(S)$-isogeny (resp. surjective homomorphism, resp. non-zero homomorphism, resp. homomorphism with κ-dimensional kernel) $A \to B$.*
   (ii) *For every m-dimensional smooth connected subscheme T of U there exists a $k(T)$-isogeny (resp. surjective homomorphism, resp. non-zero homomorphism, resp. homomorphism with κ-dimensional kernel) $A_T \to B_T$, where A_T, B_T denote the generic fibers of the base changed abelian schemes $\mathscr{A}_T \to T$, $\mathscr{B}_T \to T$, respectively (cf. Section 3).*

(b) *The following are equivalent:*
   (i) *There exists a $\overline{k(S)}$-isogeny (resp. surjective homomorphism, resp. non-zero homomorphism, resp. homomorphism with κ-dimensional kernel) $A_{\overline{k(S)}} \to B_{\overline{k(S)}}$.*
   (ii) *For every m-dimensional smooth connected subscheme T of U there exists a $\overline{k(T)}$-isogeny (resp. surjective homomorphism, resp. non-zero homomorphism, resp. homomorphism with κ-dimensional kernel) $A_{T,\overline{k(T)}} \to B_{T,\overline{k(T)}}$.*

Ingredients of the proof of Theorem A include standard methods based on the Tate conjecture (proven by Tate, Zarhin and Faltings cf. Theorem 2.3) and some consequences of the Hilbert irreducibility theorem (cf. Lemma 4.1), which were inspired by Drinfeld's "conventional formulation of Hilbertianity" in [6, Section A.1.] and by Section 2 of a recent paper of Cadoret and Tamagawa [4]. As a formal consequence we obtain:

Corollary B (Cor. 4.6): *Let S be a smooth variety over a finitely generated field k. Let $\mathscr{A}$ be an abelian scheme over S with generic fiber A. Let U be a dense open subscheme of S. Let $m \in \{0, 1, \dots, \dim(S)\}$. Assume that k is infinite or that $m \geq 1$.*

(a) *The following are equivalent:*
   (i) *A is not a simple $k(S)$-variety.*
   (ii) *For every m-dimensional smooth connected subscheme T of U the fiber A_T is not a simple $k(T)$-variety.*

(b) *The following are equivalent:*
   (i) *$A_{\overline{k(S)}}$ is not a simple $\overline{k(S)}$-variety.*
   (ii) *For every m-dimensional smooth connected subscheme T of U the fiber $A_{T,\overline{k(T)}}$ is not a simple $\overline{k(T)}$-variety.*

Our second main result is the following local to global principle for quadratic isogeny twists of abelian varieties. We call an abelian variety B/k a **quadratic isogeny twist** of an abelian variety A/k, if there exists a quadratic twist A'/k of A and a K-isogeny $B \to A'$ (cf. Section 2).

Theorem C (Thm. 4.7): *Let S be a smooth variety over a finitely generated field k. Let $\mathscr{A}, \mathscr{B}$ be abelian schemes over S with generic fibers A and B respectively. Let U be a dense open subscheme of S. Let $m \in \{0, 1, \dots, \dim(S)\}$. Assume that k is infinite or that $m \geq 1$. The following are equivalent:*

(a) *A is a quadratic isogeny twist of B*
(b) *For every m-dimensional smooth connected subscheme T of U the abelian variety A_T is a quadratic isogeny twist of B_T.*

The implication (a)$\Rightarrow$(b) holds true also in the case where k is finite and $m = 0$.

We remark, that results in Section 4 of the paper are a bit more general than Theorem A, Corollary B and Theorem C in that they also cover the situation where S is an arithmetic scheme, but we do not go into the details within this introduction.

It is clear that in the above statements the case when k is a finite field and m=0 can not be covered by the Hilbertianity methods. It constitutes a separate question which we address under two sets of additional assumptions. If A and B do not have nontrivial endomorphisms geometrically, then we establish a global function field analogue (cf. Proposition 5.8 below) of Fité's result [9, Cor. 2.7] following its proof quite closely. By combining Proposition 5.8 with the Hilbertianity approach we augment Theorem C by the case k finite and m=0.

Theorem D (Cor. 5.9): *Let S be a smooth variety over a finite field k. Let $\mathscr{A}, \mathscr{B}$ be abelian schemes over S with generic fibers A and B respectively. Let U be a dense open subscheme of S. Assume that*

$$\mathrm{End}_{\overline{k(S)}}(A) = \mathrm{End}_{\overline{k(S)}}(B) = \mathbb{Z}.$$

The following are equivalent:

(a) *A is a quadratic isogeny twist of B.*
(b) *For every closed point u of U the abelian variety A_u is a quadratic isogeny twist of B_u.*

If abelian varieties A and B meet the so-called *minuscule weights conjecture* of Zarhin (cf. condition MWC, Definition 5.1), then we apply a global function field analogue (cf. Proposition 5.4) of a result of Khare and Larsen [13, Thm. 1] and prove the following result. It completes part (b) of Theorem A in case when k is finite and m=0. We discuss the current status of Zarhin's conjecture in Remark 5.3. In particular, it holds true for ordinary abelian varieties over global fields of positive characteristics.

Theorem E (Thm. 5.6): *Let S be a smooth variety over a finite field k. Let $\mathscr{A}, \mathscr{B}$ be abelian schemes over S with generic fibers A and B respectively. Assume that A satisfies $MWC(A)$ and B satisfies $MWC(B)$. The following are equivalent:*

(a) *There exists a surjective $\overline{k(S)}$-homomorphism (resp. $\overline{k(S)}$-isogeny)*

$$A_{\overline{k(S)}} \to B_{\overline{k(S)}}.$$

(b) *For every closed point $s \in S$ there exists a surjective $\overline{k(s)}$-homomorphism (resp. $\overline{k(s)}$-isogeny)*

$$A_{s,\overline{k(s)}} \to B_{s,\overline{k(s)}}.$$

Structure of the paper. In Sections 2 and 3 we gathered material which is needed in the sequel including basic facts on: twists of abelian varieties, Galois representations and abelian schemes. Section 4 is a central part of the paper. It contains proofs of main results by Hilbertianity methods in the case when k is an infinite field or $m \geq 1$. In the final section we discuss the remaining case of k finite, m=0 and work under extra assumptions, either the minuscule weights conjecture or trivial endomorphisms for generic fibers.

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2. Preliminaries

Notation. For a field K we denote by $\overline{K}$ a separable closure of K. If E/K is a Galois extension, we denote by $\mathrm{Gal}(E/K)$ its Galois group and define

$$\mathrm{Gal}(K) := \mathrm{Gal}(\overline{K}/K).$$

A K-variety is a separated algebraic K-scheme which is reduced and irreducible. A K-curve is a K-variety of dimension 1. For a scheme S and $s \in S$ we denote by $k(s)$ the residue field of s. Let $n \in \mathbb{Z}$. Then, as usual, we denote by $S[n^{-1}]$ the open subscheme of S with underlying set $\{s \in S : n \in k(s)^\times\}$ (where n is viewed as an element of $k(s)$ via the ring homomorphism $\mathbb{Z} \to k(s)$). We let $\mathbb{L}$ be the set of all rational primes and define

$$\mathbb{L}(S) := \{\ell \in \mathbb{L} : S[\ell^{-1}] \neq \emptyset\}.$$

If S is reduced and irreducible we denote, following EGA, by $R(S)$ the function field of S. If S is a K-variety, we sometimes write $K(S)$ instead of $R(S)$. If T is a finite free $\mathbb{Z}_\ell$-module, $V = T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ and Γ is a subgroup $\mathrm{GL}_T(\mathbb{Z}_\ell)$, then we denote by Γ^{Zar} the Zariski closure of Γ inside the algebraic group $\mathrm{GL}_V/\mathbb{Q}_\ell$ so that Γ^{Zar} is an algebraic group of $\mathbb{Q}_\ell$. If $\underline{G}$ is an algebraic group over $\mathbb{Q}_\ell$, then we denote by $\underline{G}^\circ$ the connected component of the identity element of $\underline{G}$.

TWISTS OF ABELIAN VARIETIES. Let K be a field and A and B abelian varieties over K. Let E/K be a Galois extension. We call B an E/K-twist of A if there is an E-isomorphism $A_E \to B_E$. In subsequent sections we are mainly interested in the case $E = \overline{K}$. We denote by $\mathrm{Twist}_{E/K}(A)$ the set of all isomorphism classes of E/K-twists of A and define $\mathrm{Twist}(A) := \mathrm{Twist}_{\overline{K}/K}(A)$. There are natural operations $\rho_A : \mathrm{Gal}(K) \to \mathrm{Aut}_K(A_E)$ and $\rho_B : \mathrm{Gal}(K) \to \mathrm{Aut}_K(B_E)$ and an operation

$$\mathrm{Gal}(K) \times \mathrm{Hom}_E(A_E, B_E) \to \mathrm{Hom}_E(A_E, B_E),$$
$${}^{\sigma}f := \rho_B(\sigma) \circ f \circ \rho_A(\sigma)^{-1}.$$

Now assume that B/K is an E/K-twist of A and choose an E-isomorphism $f : A_E \to B_E$. Then

$$\xi : \mathrm{Gal}(E/K) \to \mathrm{Aut}_E(A_E),$$
$$\xi(\sigma) = f^{-1} \circ {}^{\sigma}f$$

is a 1-cocycle (i.e., $\xi(\sigma\eta) = \xi(\sigma) \circ {}^{\sigma}\xi(\eta)$) whose cohomology class does not depend on the choice of f. Let

$$\tilde{\rho}_B : \mathrm{Gal}(E/K) \to \mathrm{Aut}_K(A_E),$$
$$\tilde{\rho}_B(\sigma) = f^{-1} \circ \rho_B(\sigma) \circ f$$

be the operation on A derived from ρ_B via transport of structure via f. Then an easy calculation shows that

$$\tilde{\rho}_B(\sigma) = \xi(\sigma)\rho_A(\sigma) \quad \forall \sigma \in \mathrm{Gal}(E/K) \tag{1}$$

is simply the action ρ_A twisted by the 1-cocycle ξ. It is well-known that this sets up a bijective map

$$\Xi_{E/K} : \mathrm{Twist}_{E/K}(A) \to H^1(\mathrm{Gal}(E/K), \mathrm{Aut}_E(A_E)).$$

There is a natural map

$$j : H^1(\mathrm{Gal}(E/K), \{\pm \mathrm{Id}_A\}) \to H^1(\mathrm{Gal}(E/K), \mathrm{Aut}_E(A_E)).$$

For a quadratic character $\chi \in \mathrm{Hom}(\mathrm{Gal}(E/K), \{\pm\mathrm{Id}_A\})$ we denote by A_χ an E/K-twist of A corresponding to $j(\chi)$ under $\Xi_{E/K}$. We call B **a quadratic isogeny twist of** A, if there exists a quadratic character

$$\chi \in \mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), \{\pm\mathrm{Id}_A\})$$

and a K-isogeny

$$B \to A_\chi.$$

Sometimes we tacitly identify $\{\pm\mathrm{Id}_A\}$ with $\mu_2(\mathbb{Q}) = \{\pm 1\}$.

Remark 2.1: Let A be an abelian variety over a finite field K. Then there is a unique non-trivial quadratic character $\mathrm{Gal}(K) \to \{\pm \mathrm{Id}_A\}$ and accordingly a unique non-trivial quadratic twist of A. This is clear since $\mathrm{Gal}(K) \cong \hat{\mathbb{Z}}$.

Galois representations. Let K be a field of characteristic $p \geq 0$ and A/K an abelian variety. For a rational prime $\ell \neq p$ we denote by

$$\rho_{A,\ell} : \mathrm{Gal}(K) \to \mathrm{GL}_{V_\ell(A)}(\mathbb{Q}_\ell)$$

the corresponding ℓ-adic Galois representation, where $T_\ell(A)$ is the ℓ-adic Tate module of A and $V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Then the Zariski closure $\rho_{A,\ell}(\mathrm{Gal}(K))^{\mathrm{Zar}}$ of $\rho_{A,\ell}(\mathrm{Gal}(K))$ in $\mathrm{GL}_{V_\ell(A)}$ and its identity component $(\rho_{A,\ell}(\mathrm{Gal}(K))^{\mathrm{Zar}})^\circ$ are algebraic groups over $\mathbb{Q}_\ell$.

If K is finite, then we define the L-series of A by

$$L(A/K, T) := \det(\mathrm{Id}_{V_\ell(A)} - \rho_{A,\ell}(\mathrm{Fr})T)$$

where $\mathrm{Fr} \in \mathrm{Gal}(K)$ is the Frobenius element. This characteristic polynomial $L(A/K, T)$ has integer coefficients and does not depend on the rational prime $\ell \neq p$ by the Weil conjectures. We have the following elementary but useful fact.

Lemma 2.2: *Let K be a field of characteristic $p \geq 0$ and ℓ a prime different from p. Let A/K and B/K be abelian varieties. Let $f : A \to B$ be a homomorphism and let $V_\ell(f) : V_\ell(A) \to V_\ell(B)$ be the homomorphism of $\mathbb{Q}_\ell$-vector spaces induced by f. Then:*

(a) $\dim_{\mathbb{Q}_\ell}(\mathrm{im}(V_\ell(f))) = 2\dim(\mathrm{im}(f))$.
(b) $\dim_{\mathbb{Q}_\ell}(\ker(V_\ell(f))) = 2\dim(\ker(f))$.
(c) *$f : A \to B$ is an isogeny if and only if the map $V_\ell(f)$ is bijective. In particular, we have $\rho_{A,\ell} \cong \rho_{B,\ell}$ provided B is K-isogenous to A.*

Proof. We note that $T_\ell(A) = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, A(\overline{K}))$. For the purpose of that proof we put $T_\ell(M) := \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, M)$ for an arbitrariy abelian group M. Let $C = \ker(f)$ and $I = \mathrm{im}(f)$. Then C° and I are abelian varieties. From the exact sequence

$$0 \to C^\circ(\overline{K}) \to C(\overline{K}) \to C(\overline{K})/C^\circ(\overline{K}) \to 0$$

we derive an exact sequence

$$0 \to T_\ell(C^\circ(\overline{K})) \to T_\ell(C(\overline{K})) \to T_\ell(C(\overline{K})/C^\circ(\overline{K}))$$

because the functor $\mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, -)$ is left exact. But $T_\ell(C(\overline{K})/C^\circ(\overline{K})) = 0$ because the group $C(\overline{K})/C^\circ(\overline{K})$ is finite. Thus $T_\ell(C^\circ(\overline{K})) = T_\ell(C(\overline{K}))$. From the exact sequence

$$0 \to C(\overline{K}) \to A(\overline{K}) \to I(\overline{K}) \to 0 \tag{2}$$

we obtain an exact sequence $0 \to T_\ell(C(\overline{K})) \to T_\ell(A(\overline{K})) \to T_\ell(I(\overline{K}))$, again because the functor $\mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, -)$ is left exact. Using $T_\ell(C^\circ(\overline{K})) = T_\ell(C(\overline{K}))$ and tensoring with $\mathbb{Q}_\ell$ we see that we have an exact sequence

$$0 \to V_\ell(C^\circ) \to V_\ell(A) \to V_\ell(I). \tag{3}$$

Let $c = \dim(C)$, $a = \dim(A)$ and $i = \dim(I)$. Then $a = i + c$ by (2). Also $\dim_{\mathbb{Q}_\ell}(V_\ell(C)) = 2c$, $\dim_{\mathbb{Q}_\ell}(V_\ell(A)) = 2a$ and $\dim_{\mathbb{Q}_\ell}(V_\ell(I)) = 2i$ (cf. [15, Remark 8.4]). The image of the right hand map of the exact sequence (3) has dimension $2a - 2c = 2i$, too, hence $V_\ell(A) \to V_\ell(I)$ is surjective. By the left exactness of $\mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, -)$ we also see that $V_\ell(I) \to V_\ell(B)$ is injective. Thus $V_\ell(I) = \mathrm{im}(V_\ell(f))$ and $V_\ell(C^\circ) = \ker(V_\ell(f))$. It follows that

$$\begin{aligned}
\dim_{\mathbb{Q}_\ell}(\ker(V_\ell(f))) &= \dim_{\mathbb{Q}_\ell}(V_\ell(C^\circ)) = 2\dim(C^\circ) = 2\dim(C) = 2\dim(\ker(f)),\\
\dim_{\mathbb{Q}_\ell}(\mathrm{im}(V_\ell(f))) &= \dim_{\mathbb{Q}_\ell}(V_\ell(I)) = 2\dim(I) = 2\dim(\mathrm{im}(f)),
\end{aligned}$$

as desired. This finishes the proof of (a) and (b). For (c) put $b := \dim(B)$ and note that:

$$\begin{aligned}
f \text{ is an isogeny} &\Leftrightarrow f \text{ is surjective with a finite kernel}\\
&\Leftrightarrow \dim(\ker(f)) = 0 \text{ and } \dim(\mathrm{im}(f)) = b\\
&\Leftrightarrow \dim_{\mathbb{Q}_\ell}(\ker(V_\ell(f))) = 0 \text{ and } \dim_{\mathbb{Q}_\ell}(\mathrm{im}(V_\ell(f))) = 2b\\
&\Leftrightarrow V_\ell(f) \text{ is bijective.} \qquad \blacksquare
\end{aligned}$$

There is the following celebrated result due to Faltings, Tate and Zarhin.

THEOREM 2.3 (cf. [7], [22], [23]): *Let K be a finitely generated field of characteristic $p \geq 0$ and A/K and B/K be abelian varieties.*

(a) *$\rho_{A,\ell}$ is semisimple for every rational prime $\ell \neq p$.*
(b) *The natural homomorphism*

$$\mathrm{Hom}_K(A, B) \otimes \mathbb{Z}_\ell \to \mathrm{Hom}_{\mathrm{Gal}(K)}(T_\ell(A), T_\ell(B))$$

is bijective.

COROLLARY 2.4: *Let K be a finitely generated field of characteristic $p \geq 0$ and A/K and B/K be abelian varieties. The following statements are equivalent:*

(a) *There exists a K-isogeny $f : A \to B$.*
(b) *There exists an isomorphism $\rho_{A,\ell} \cong \rho_{B,\ell}$ of $\mathbb{Q}_\ell$-representations of* $\mathrm{Gal}(K)$*.*

If K is finite, then conditions (a) and (b) are also equivalent to the following condition:

(c) $L(A/K, T) = L(B/K, T)$.

Proof. The implication (a)⇒(b) is Lemma 2.2. The implication (b)⇒(a) is a consequence of Theorem 2.3, cf. [7, Korollar 2]. If K is finite, then the equivalence (b)⇔(c) has been established by Tate [22, Theorem 1]. ∎

Remark 2.5: Let K be a finitely generated field of characteristic $p \geq 0$ and A/K an abelian variety. Let $\chi : \mathrm{Gal}(K) \to \{\pm \mathrm{Id}_A\}$ be a quadratic character. Then:

(a) $\rho_{A_\chi,\ell} \cong \chi \otimes \rho_{A,\ell}$.
(b) If K is finite and χ is the (unique) non-trivial quadratic character of $\mathrm{Gal}(K)$, then $L(A_\chi/K, T) = L(A/K, -T)$.

Proof. (a) is immediate from equation (1). Part (b) is an immediate consequence thereof. ∎

LEMMA 2.6: *Let K be a finitely generated field of characteristic $p \geq 0$. Let A, B be abelian varieties over K and $\ell \neq p$ a rational prime. Then the following statements are equivalent:*

(a) *B is a quadratic isogeny twist of A.*
(b) *There exists a quadratic character $\chi : \mathrm{Gal}(K) \to \{\pm \mathrm{Id}_A\}$ such that $\rho_{B,\ell} \cong \chi \otimes \rho_{A,\ell}$.*

If K is finite, then the equivalent conditions (a) and (b) are also equivalent to

(c) $L(B/K, T) = L(A/K, T)$ *or* $L(B/K, T) = L(A/K, -T)$.

Proof. We prove the implication (a)⇒(b): Assume B is a quadratic isogeny twist of A. Then there exists a quadratic character $\chi : \mathrm{Gal}(K) \to \{\pm \mathrm{Id}_A\}$ and a K-isogeny $B \to A_\chi$. From Lemma 2.2 and Remark 2.5 we conclude that

$$\rho_{B,\ell} \cong \rho_{A_\chi,\ell} \cong \chi \otimes \rho_{A,\ell}.$$

For the proof of the implication (b)⇒(a) assume that there exists a quadratic character $\chi : \mathrm{Gal}(K) \to \{\pm\mathrm{Id}_A\}$ such that $\rho_{B,\ell} \cong \chi \otimes \rho_{A,\ell}$. From Remark 2.5 we conclude that $\rho_{B,\ell} \cong \rho_{A_\chi,\ell}$ and then Corollary 2.4 implies that there exists an isogeny $B \to A_\chi$. It follows that B is a quadratic isogeny twist of A.

From now on assume that K is finite. Then $\mathrm{Gal}(K) = \hat{\mathbb{Z}}$ and thus there exist exactly two quadratic characters thereof.

We prove the implication (b)⇒(c): If χ is the trivial character then $\rho_{B,\ell} \cong \rho_{A,\ell}$ and hence $L(A/K,T) = L(B/K,T)$. If χ is non-trivial, then $\rho_{B,\ell} \cong \rho_{A_\chi,\ell}$, hence

$$L(B/K,T) = L(A_\chi,T) = L(A/K,-T)$$

by Corollary 2.4 and Remark 2.5.

We prove the implication (c)⇒(a): If $L(B/K,T) = L(A/K,T)$, then A is isogenuous to B by Corollary 2.4. If $L(B/K,T) = L(A/K,-T)$ and χ the nontrivial quadratic character, then

$$L(B/K,T) = L(A_\chi/K,T)$$

and Corollary 2.4 implies that B is isogenous to A_χ, as desired. ∎

Remark 2.7: Let A and B be abelian varieties over a field K and ℓ a prime different from the characteristic of K. Note that

$$\rho_{A\times B,\ell}(\sigma) = \rho_{A,\ell}(\sigma) \times \rho_{B,\ell}(\sigma) \quad \text{for all } \sigma \in \mathrm{Gal}(K).$$

Thus

$$\rho_{A\times B,\ell}(\mathrm{Gal}(K)) \subset \mathrm{GL}_{V_\ell(A)}(\mathbb{Q}_\ell) \times \mathrm{GL}_{V_\ell(B)}(\mathbb{Q}_\ell) \subset \mathrm{GL}_{V_\ell(A)\times V_\ell(B)}(\mathbb{Q}_\ell).$$

Let p_A and p_B be the projections of the product $\mathrm{GL}_{V_\ell(A)}(\mathbb{Q}_\ell) \times \mathrm{GL}_{V_\ell(B)}(\mathbb{Q}_\ell)$. Then

$$\rho_{A,\ell}(\sigma) = p_A(\rho_{A\times B,\ell}(\sigma)) \quad \text{and} \quad \rho_{B,\ell}(\sigma) = p_B(\rho_{A\times B,\ell}(\sigma))$$

for all $\sigma \in \mathrm{Gal}(K)$. Thus the actions of $\mathrm{Gal}(K)$ on $V_\ell(A)$ and $V_\ell(B)$ and the induced action on $\mathrm{Hom}_{\mathbb{Q}_\ell}(V_\ell(A), V_\ell(B))$ factor through $G = \rho_{A\times B,\ell}(\mathrm{Gal}(K))$. Let $\underline{G} = \rho_{A\times B,\ell}(\mathrm{Gal}(K))^{\mathrm{Zar}}$. If K is finitely generated, then the natural map

$$\mathrm{Hom}_K(A,B) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to \mathrm{Hom}_{\mathbb{Q}_\ell}(V_\ell(A), V_\ell(B))^G = \mathrm{Hom}_{\mathbb{Q}_\ell}(V_\ell(A), V_\ell(B))^{\underline{G}} \tag{4}$$

is bijective by the Tate conjecture (cf. Theorem 2.3).

The following lemma and its proof are inspired by [20, Prop 2.10].

Lemma 2.8: *Let A and B be abelian varieties over a field K and keep the notation from Remark 2.7. Let K° be the fixed field of the group $(\rho_{A\times B,\ell})^{-1}(\underline{G}^\circ)$. Then*

$$\mathrm{Hom}_{K^\circ}(A_{K^\circ}, B_{K^\circ}) = \mathrm{Hom}_{\overline{K}}(A_{\overline{K}}, B_{\overline{K}}). \tag{5}$$

If K is finitely generated, then the above specialization map (4) *induces a bijective map*

$$\mathrm{Hom}_{\overline{K}}(A_{\overline{K}}, B_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to \mathrm{Hom}_{\mathbb{Q}_\ell}(V_\ell(A), V_\ell(B))^{\underline{G}^\circ}. \tag{6}$$

Proof. For (5) it is enough to prove that

$$\mathrm{Hom}_{K^\circ}(A_{K^\circ}, B_{K^\circ}) = \mathrm{Hom}_{K'}(A_{K'}, B_{K'})$$

for every finite Galois extension K'/K with $K^\circ \subset K'$. The profinite group $\mathrm{Gal}(K')$ is of finite index in $\mathrm{Gal}(K)$ and thus

$$\rho_{A\times B,\ell}(\mathrm{Gal}(K'))^{\mathrm{Zar}}$$

is a closed subgroup scheme of finite index in $\underline{G}^\circ$. Thus finitely many cosets of $\rho_{A\times B,\ell}(\mathrm{Gal}(K'))^{\mathrm{Zar}}$ cover $\underline{G}^\circ$. As $\underline{G}^\circ$ is connected, it follows that

$$\rho_{A\times B,\ell}(\mathrm{Gal}(K'))^{\mathrm{Zar}} = \underline{G}^\circ.$$

Let $H = \mathrm{Hom}_{\mathbb{Q}_\ell}(V_\ell(A), V_\ell(B))$ and consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{K^\circ}(A_{K^\circ}, B_{K^\circ}) \arrow[r] \arrow[d] & H^{\underline{G}^\circ} \arrow[r, equal] \arrow[d, "\mathrm{Id}"'] & H^{\mathrm{Gal}(K^\circ)} \arrow[d, "\mathrm{Id}"'] \\
\mathrm{Hom}_{K'}(A_{K'}, B_{K'}) \arrow[r] & H^{\underline{G}^\circ} \arrow[r, equal] & H^{\mathrm{Gal}(K')}
\end{tikzcd}$$

The horizontal arrows are injective (cf. Theorem [15, Prop. 12.2]). Thus, for every $f \in \mathrm{Hom}_{K'}(A_{K'}, B_{K'})$, the map $V_\ell(f) \in H$ is invariant under $\mathrm{Gal}(K^\circ)$ and, by the injectivity of the horizontal maps of the diagram, f itself is invariant under $\mathrm{Gal}(K^\circ)$ which implies $f \in \mathrm{Hom}_{K^\circ}(A_{K^\circ}, B_{K^\circ})$. This finishes up the proof of (5). If K is finitely generated, then (5) together with the Tate conjecture (cf. Theorem 2.3) implies that (6) is bijective. ∎

3. Abelian schemes

Throughout this section let S and T be noetherian, normal and connected schemes. Note that then the local rings of S are domains and thus [10, 6.1.10] (together with the connectedness of S) implies that S is in fact irreducible. Furthermore S is reduced. Similarly T is reduced and irreducible. Let F (resp. E) be the function field of S (resp. T). Let $u : T \to S$ be a morphism. Furthermore consider the point

$$t : \mathrm{Spec}(E) \to T \xrightarrow{u} S$$

of S. We fix throughout this section a rational prime $\ell \in \mathbb{L}(T)$. This is possible by the following lemma. We note that automatically $\ell \in \mathbb{L}(S)$.

Lemma 3.1: *For every scheme X, the set $\mathbb{L}(X)$ is non-empty.*

Proof. If there exists $p \in \mathbb{L} \cup \{0\}$ such that $\mathrm{char}(k(x)) = p$ for all $x \in X$, then $\mathbb{L}(X) = \mathbb{L} \setminus \{p\}$ is not empty. Otherwise, there exist $x_1, x_2 \in X$ such that $\mathrm{char}(k(x_1)) \neq \mathrm{char}(k(x_2))$ and $\mathrm{char}(k(x_1)) > 0$, hence putting

$$\ell := \mathrm{char}(k(x_1))$$

we get $x_2 \in X[\ell^{-1}]$ and thus $\ell \in \mathbb{L}(X)$, so that $\mathbb{L}(X)$ is non-empty also in that case. ∎

Let $\mathscr{A}/S$ be an abelian scheme with generic fiber A/F. Let $\mathscr{A}_T := \mathscr{A} \times_S T$ and let $A_T := \mathscr{A} \times_S \mathrm{Spec}(E)$ be the generic fiber of $\mathscr{A}_T \to T$. We then have cartesian squares

$$\begin{tikzcd}
A_T \arrow[r] \arrow[d] & \mathscr{A}_T \arrow[r] \arrow[d] & \mathscr{A} \arrow[d] & A \arrow[l] \arrow[d] \\
\mathrm{Spec}(E) \arrow[r] & T \arrow[r, "u"] & S & \mathrm{Spec}(F) \arrow[l]
\end{tikzcd}$$

Let $n \in \mathbb{N}$. Then the restriction of $\mathscr{A}[\ell^n]$ to $S[\ell^{-1}]$ is a finite étale $S[\ell^{-1}]$-scheme of rank $\ell^{2\dim(A)n}$. In particular, the action of $\mathrm{Gal}(F)$ on $A[\ell^n](\overline{F})$ factors through $\pi_1(S[\ell^{-1}])$. Likewise $\rho_{A,\ell}$ factors through $\pi_1(S[\ell^{-1}])$, i.e., we can view it as a homomorphism

$$\rho_{A,\ell} : \pi_1(S[\ell^{-1}]) \to GL_{V_\ell(A)}(\mathbb{Q}_\ell).$$

There exists a finite connected étale cover $S' \to S[\ell^{-1}]$ such that $\mathscr{A}[\ell^n] \times S'$ splits up in a coproduct of $\ell^{2n\dim(A)}$ copies of S', i.e., it is a constant group scheme. There is a point $t' : \mathrm{Spec}(\overline{E}) \to S'$ over t and a point $\xi' : \overline{F} \to S'$ lying over the generic point of S. As $\mathscr{A}[\ell^n] \times_S S'$ is a constant group scheme, the natural maps

$$A[\ell^n](\overline{F}) \leftarrow \mathscr{A}[\ell^n](S') \to A_T[\ell^n](\overline{E})$$

derived from these points are bijective. We thus get a specialization isomorphism

$$s_{A,T,\ell^n} : A[\ell^n](\overline{F}) \to A_T[\ell^n](\overline{E})$$

and accordingly a specialization isomorphism

$$s_{A,T,\ell^\infty} : T_\ell(A) \to T_\ell(A_T).$$

These are equivariant for the action of $\pi_1(S[\ell^{-1}])$. In particular we have

$$\dim(A) = \dim(A_T).$$

Definition 3.2: Let G be a group, S_0 an open subscheme of S and $\rho : \pi_1(S_0) \to G$ a homomorphism. Let $T_0 := u^{-1}(S_0)$. We define $u^*\rho$ to be the composite morphism[1]

$$\pi_1(T_0) \xrightarrow{u_*} \pi_1(S_0) \to G.$$

If u is clear from the context we put $\rho_T := u^*\rho$. We say that T (or u) is **ρ-generic** if T_0 is not empty and

$$\rho_T(\pi_1(T_0)) = \rho(\pi_1(S_0)).$$

We often apply this notation in the case where $S_0 = S[\ell^{-1}]$ and $\rho = \rho_{A,\ell}$. Note that in that case $u^{-1}(S_0) = T[\ell^{-1}]$ is automatically non-empty by our choice of ℓ.

Remark 3.3: The specialization isomorphism $s_{A,T,\ell^\infty} : T_\ell(A) \to T_\ell(A_T)$ gives an isomorphism

$$u^*\rho_{A,\ell} \cong \rho_{A_T,\ell}.$$

of representations of $\mathrm{Gal}(E)$.

For the rest of this section let $\mathscr{B}$ be an abelian scheme over S with generic fiber B/F. Define $\mathscr{B}_T := \mathscr{B} \times_S T$ and let $B_T := \mathscr{B} \times_T \mathrm{Spec}(E)$ be the generic fiber of $\mathscr{B}_T \to T$.

[1] Note: The fundamental group of the empty scheme is the trivial group.

LEMMA 3.4: *The canonical map*

$$\mathrm{Hom}_S(\mathscr{A}, \mathscr{B}) \to \mathrm{Hom}_F(A, B), \;\; f \mapsto f_F$$

is bijective.

Proof. Let $\mathscr{U}$ be the category of all dense affine open subschemes of S. Then

$$\mathrm{Spec}(F) = \varprojlim_{U \in \mathscr{U}} U$$

(projective limit of schemes, cf. [11, 8.2]). It follows from [11, 8.8.2] that the natural map

$$\varinjlim_{U \in \mathscr{U}} \mathrm{Hom}_U(\mathscr{A}_U, \mathscr{B}_U) \to \mathrm{Hom}_F(A, B), \tag{7}$$

induced by the natural maps $\mathrm{Hom}_U(\mathscr{A}_U, \mathscr{B}_U) \to \mathrm{Hom}_F(A, B)$, $f \mapsto f_F$, is bijective. Moreover, by [8, Prop. 2.7], for every $U \in \mathscr{U}$ the natural map

$$\mathrm{Hom}_S(\mathscr{A}, \mathscr{B}) \to \mathrm{Hom}_U(\mathscr{A}_U, \mathscr{B}_U), \;\; f \mapsto f_U$$

is bijective, and therefore the natural map

$$\mathrm{Hom}_S(\mathscr{A}, \mathscr{B}) \to \varinjlim_{U \in \mathscr{U}} \mathrm{Hom}_U(\mathscr{A}_U, \mathscr{B}_U) \tag{8}$$

is bijective. The assertion is immediate from the bijectivity of (7) and (8). ∎

LEMMA 3.5: *The canonical map*

$$\mathrm{Hom}_S(\mathscr{A}, \mathscr{B}) \to \mathrm{Hom}_T(\mathscr{A}_T, \mathscr{B}_T) \tag{9}$$

is injective.

Proof. Choose a point $t \in T$. We even prove that the composite homomorphism

$$\mathrm{Hom}_S(\mathscr{A}, \mathscr{B}) \to \mathrm{Hom}_T(\mathscr{A}_T, \mathscr{B}_T) \to \mathrm{Hom}_{k(t)}(\mathscr{A}_t, \mathscr{B}_t) \tag{10}$$

is injective. Let f be in the kernel of (10). Then

$$f_t : \mathscr{A} \times_S \mathrm{Spec}(k(t)) \to \mathscr{B} \times_S \mathrm{Spec}(k(t))$$

is the zero homomorphism. Hence f must be the zero homomorphism by the rigidity lemma [15, 20.1]. ∎

Note that, by Lemmata 3.4 and 3.5 above, there is a canonical injective specialization map

$$\begin{aligned} r_{\mathscr{A},\mathscr{B},T,S} : \mathrm{Hom}_F(A,B) \cong & \mathrm{Hom}_S(\mathscr{A},\mathscr{B}) \\ & \longrightarrow \mathrm{Hom}_T(\mathscr{A}_T,\mathscr{B}_T) \cong \mathrm{Hom}_E(A_T,B_T). \end{aligned} \tag{11}$$

Let $\rho_\ell := \rho_{A\times B,\ell}$,

$$\underline{G} := \rho_\ell(\pi_1(S[\ell^{-1}]))^{\mathrm{Zar}}$$

and let S' be the finite étale cover of $S[\ell^{-1}]$ corresponding to the subgroup $\rho_\ell^{-1}(\underline{G}^\circ)$ of $\pi_1(S[\ell^{-1}])$. Let F' be the function field of S'. Then, by Lemma 2.8,

$$\mathrm{Hom}_{F'}(A_{F'},B_{F'}) = \mathrm{Hom}_{\overline{F}}(A_{\overline{F}},B_{\overline{F}}).$$

Let T' be an irreducible component of $T\times_S S'$ and E' the function field of T'. Then E'/E is a finite separable extension and we can consider the composite map

$$\begin{aligned} \overline{r}_{\mathscr{A},\mathscr{B},T,S} : & \mathrm{Hom}_{\overline{F}}(A_{\overline{F}},B_{\overline{F}}) = \mathrm{Hom}_{F'}(A_{F'},B_{F'}) \\ & \xrightarrow{r_{\mathscr{A},\mathscr{B},T',S'}} \mathrm{Hom}_{E'}(A_{T,E'},B_{T,E'}) \to \mathrm{Hom}_{\overline{E}}(A_{T,\overline{E}},B_{T,\overline{E}}) \end{aligned} \tag{12}$$

which tacitly depends on the choice of T' and on the choice of an embedding $E'\to\overline{E}$. This map

$$\overline{r}_{\mathscr{A},\mathscr{B},T,S} : \mathrm{Hom}_{\overline{F}}(A_{\overline{F}},B_{\overline{F}}) \to \mathrm{Hom}_{\overline{E}}(A_{T,\overline{E}},B_{T,\overline{E}})$$

is injective.

Lemma 3.6: *Let $f : A\to B$ be a homomorphism. Let $f_T : A_T \to B_T$ be the homomorphism $f_T := r_{\mathscr{A},\mathscr{B},T,S}(f)$ obtained from f by specialization.*

(a) $\dim(\mathrm{im}(f)) = \dim(\mathrm{im}(f_T))$,
(b) $\dim(\ker(f)) = \dim(\ker(f_T))$,
(c) *f has finite kernel (resp. is surjective, resp. is an isogeny) if and only if f_T has finite kernel (resp. is surjective, resp. is an isogeny).*

Proof. With the above specialization isomorphisms we construct a diagram

$$\begin{array}{ccc} V_\ell(A) & \xrightarrow{V_\ell(f)} & V_\ell(B) \\ \downarrow & & \downarrow \\ V_\ell(A_T) & \xrightarrow{V_\ell(f_T)} & V_\ell(B_T) \end{array}$$

whose vertical arrows are bijective. Together with Lemma 2.2 we get:

$$\dim(\ker(f)) = \frac{1}{2}\dim(\ker(V_\ell(f))) = \frac{1}{2}\dim(\ker(V_\ell(f_T))) = \dim(\ker(f_T)),$$
$$\dim(\mathrm{im}(f)) = \frac{1}{2}\dim(\mathrm{im}(V_\ell(f))) = \frac{1}{2}\dim(\mathrm{im}(V_\ell(f_T))) = \dim(\mathrm{im}(f_T)).$$

This proves (a) and (b), and (c) is immediate from that. ∎

Remark 3.7: For $f \in \mathrm{Hom}_{\overline{F}}(A_{\overline{F}}, B_{\overline{F}})$ and $f_T := \overline{r}_{\mathscr{A},\mathscr{B},T,S}(f)$ one can compare dimension data of f and f_T in a completely analogous way. This is a consequence of Lemma 3.6 and the construction of $\overline{r}_{\mathscr{A},\mathscr{B},T,S}$.

Lemma 3.8: *If both fields E and F are finitely generated and $u : T \to S$ is $\rho_{A\times B,\ell}$-generic, then the canonical maps*

$$r_{\mathscr{A},\mathscr{B},T,S} \otimes \mathbb{Z}_\ell : \mathrm{Hom}_F(A, B) \otimes \mathbb{Z}_\ell \to \mathrm{Hom}_E(A_T, B_T) \otimes \mathbb{Z}_\ell$$

and

$$\overline{r}_{\mathscr{A},\mathscr{B},T,S} \otimes \mathbb{Z}_\ell : \mathrm{Hom}_{\overline{F}}(A_{\overline{F}}, B_{\overline{F}}) \otimes \mathbb{Z}_\ell \to \mathrm{Hom}_{\overline{E}}(A_{T,\overline{E}}, B_{T,\overline{E}}) \otimes \mathbb{Z}_\ell$$

are bijective. In particular, under these assumptions $\mathrm{coker}(r_{\mathscr{A},\mathscr{B},T,S})$ *and* $\mathrm{coker}(\overline{r}_{\mathscr{A},\mathscr{B},T,S})$ *are finite groups of order prime to ℓ.*

Proof. We identify, $T_\ell(A)$ with $T_\ell(A_T)$ and $T_\ell(B)$ with $T_\ell(B_T)$ along the natural and equivariant specialization isomorphisms. Let $\rho_\ell := \rho_{A\times B,\ell}$,

$$\underline{G} := \rho_\ell(\pi_1(S[\ell^{-1}]))^{\mathrm{Zar}} \quad \text{and} \quad \underline{G}_T := (\rho_\ell)_T(\pi_1(T[\ell^{-1}]))^{\mathrm{Zar}}.$$

Let

$$r_\ell = r_{\mathscr{A},\mathscr{B},T,S} \otimes \mathbb{Z}_\ell$$

and $\overline{r_\ell} = \overline{r}_{\mathscr{A},\mathscr{B},T,S} \otimes \mathbb{Z}_\ell$. We then have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_F(A, B) \otimes \mathbb{Z}_\ell & \xrightarrow{r_\ell} & \mathrm{Hom}_E(A_T, B_T) \otimes \mathbb{Z}_\ell \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathbb{Z}_\ell}(V_\ell(A), V_\ell(B))^{\underline{G}} & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_\ell}(V_\ell(A), V_\ell(B))^{\underline{G}_T},
\end{array}$$

where the vertical maps are bijective (cf. Tate conjecture, Remark 2.7). The lower horizonal map is bijective because $\underline{G} = \underline{G}_T$ by our assumption that T is ρ_ℓ-generic. Hence the upper horizontal map r_ℓ is bijective, too.

Next consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\overline{F}}(A_{\overline{F}}, B_{\overline{F}}) \otimes \mathbb{Z}_\ell & \xrightarrow{\overline{r_\ell}} & \mathrm{Hom}_{\overline{E}}(A_{T,\overline{E}}, B_{T,\overline{E}}) \otimes \mathbb{Z}_\ell \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathbb{Z}_\ell}(V_\ell(A), V_\ell(B))^{\underline{G}^\circ} & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_\ell}(V_\ell(A), V_\ell(B))^{\underline{G}_T^\circ} \end{array}$$

where the vertical maps are bijective (cf. Tate conjecture, Lemma 2.8). The lower horizonal map is bijective because

$$\underline{G}^\circ = \underline{G}_T^\circ.$$

Hence, $\overline{r_\ell}$ is bijective. The $\mathbb{Z}$-module $\mathrm{Hom}_{\overline{E}}(A_{T,\overline{E}}, B_{T,\overline{E}})$ is free and finitely generated. Hence, the statement about the cokernels of $r_{\mathscr{A},\mathscr{B},T,S}$ and $\overline{r}_{\mathscr{A},\mathscr{B},T,S}$ follows as well. ∎

4. Hilbertianity

Throughout this section let Z be a regular noetherian connected scheme. Let S be a connected scheme and $f : S \to Z$ a morphism of finite type which is assumed to be smooth. Note that S and Z are reduced and irreducible. Let d be the relative dimension of S/Z. Let K be the function field of Z. Assume that K is finitely generated. We denote by $\mathrm{Sm}_m(S/Z)$ the set of all connected subschemes T of S such that the restriction $f|T : T \to Z$ is smooth of relative dimension m. Note that every $T \in \mathrm{Sm}_m(S/Z)$ is regular and connected, hence reduced and irrecucible.

The aim of this section is to consider specializations to subvarieties in $\mathrm{Sm}_m(S/Z)$. The results of this section are in our opinion most interesting in the following cases:

(1) The case where $Z = \mathrm{Spec}(k)$ for a finitely generated field k; in that case S is simply a smooth k-variety.
(2) The case where Z is an open subscheme of $\mathrm{Spec}(R)$ and R is the ring of integers in a number field; in that case S is sometimes called an **arithmetic scheme**.

The following lemma is inspired by Drinfeld's "conventional formulation of Hilbertianity" in [6, Section A.1.].

Lemma 4.1: *If K is Hilbertian, then for every dense open subscheme U of S and every finite étale morphism $p : X \to U$ there exists a connected subscheme T of U with the following properties.*

(a) *T is a subscheme of $(f|U)^{-1}(Z')$ for some dense open subscheme Z' of Z.*
(b) *$f|T : T \to Z'$ is finite and étale.*
(c) *$p^{-1}(T) = X \times_S T$ is connected.*

Proof. As f is smooth, after replacing U by a smaller dense open set and after replacing X accordingly, there exists an étale Z-morphism $g : U \to Z \times \mathbb{A}_d$ (cf. [12, Exposé II, §1]). For any dense open subscheme Z' of Z we can consider the following commutative diagram of schemes

$$\begin{tikzcd} X_K \arrow[r, "p_K"] \arrow[d] & U_K \arrow[r, "g_K"] \arrow[d] & \mathbb{A}_{d,K} \arrow[r] \arrow[d] & Spec(K) \arrow[d] \\ X' \arrow[r, "p'"] & U' \arrow[r, "g'"] & Z' \times \mathbb{A}_d \arrow[r] & Z', \end{tikzcd}$$

where $U' = U \times_Z Z'$, $X' = X \times_Z Z'$ and g' and p' are the restrictions of g and p respectively. As K is Hilbertian there is a point $a \in \mathbb{A}_d(K)$ such that

$$(g_K \circ p_K)^{-1}(a) = \mathrm{Spec}(F) \quad \text{and} \quad g_K^{-1}(a) = \mathrm{Spec}(E)$$

where E/K and F/E are finite separable field extensions. For a suitable choice of Z' the closed immersion $a : \mathrm{Spec}(K) \to \mathbb{A}_{d,K}$ extends to a closed subscheme Y of $Z' \times \mathbb{A}_d$ (cf. [11, 8.8.2 and 8.10.5]). Put $T := g^{-1}(Y)$ and $X_T := p^{-1}(T)$. After replacing Z' by one of its dense open subschemes we can assume that the maps $p|X_T : X_T \to T$ and $g : T \to Y$ are finite and étale, because the corresponding maps on the generic fibers

$$(g_K \circ p_K)^{-1}(a) = \mathrm{Spec}(F) \to g_K^{-1}(a) = \mathrm{Spec}(E) \to \mathrm{Spec}(K)$$

are finite and étale. The closed subscheme $T := g^{-1}(Y)$ of U' is connected because it is finite and étale over Z' and its generic fiber

$$T \times_{Z'} \mathrm{Spec}(K) = g_K^{-1}(a) = \mathrm{Spec}(E)$$

is connected. Likewise the closed subscheme X_T of X' is connected, because it is finite and étale over Z' and its generic fiber $X_T \times_{Z'} \mathrm{Spec}(K) = \mathrm{Spec}(F)$ is connected. ∎

Corollary 4.2: *Let G be a profinite group and assume that the Frattini subgroup $\Phi(G)$ of G is open in G. Let $\rho : \pi_1(S) \to G$ be a group homomorphism. For every dense open subscheme U of S and every $m \in \{1, \ldots, d\}$ there exists a $T \in \mathrm{Sm}_m(S/Z)$ such that $T \subset U$ and such that T is ρ-generic. If K is Hilbertian, then the same holds true for $m = 0$.*

Proof. We consider the homomorphism

$$\rho' : \pi_1(S) \xrightarrow{\rho} G \to G/\Phi(G).$$

The image of ρ' is finite because $\Phi(G)$ is open in G. By the Frattini property, a subscheme $T \in \mathrm{Sm}_m(S/Z)$ is ρ-generic if and only if it is ρ'-generic. In the proof of the corollary we can hence assume that G is finite.

Case A: Assume that K is Hilbertian and $m = 0$. Let X be the finite étale cover of S corresponding to the kernel of ρ. By Lemma 4.1 there exists $T \in \mathrm{Sm}_m(S/Z)$ such that $T \subset U$ and such that $T \times_S X$ is connected. This implies that T is ρ-generic.

Case B: Assume that $m \in \{1, 2, \ldots, d\}$ (and K arbitrary). As $f : S \to Z$ is smooth we can, after replacing U by a smaller open set, assume that there exists an étale Z-morphism $U \to Z \times \mathbb{A}_d$. Composing with an appropriate projection we get a smooth morphism

$$S \to Z \times \mathbb{A}_d \to Z \times \mathbb{A}_m.$$

Note that $\mathrm{Sm}_0(S/Z \times \mathbb{A}_m) \subset \mathrm{Sm}_m(S/Z)$. The function field $K(x_1, \ldots, x_m)$ of $Z \times \mathbb{A}_m$ is Hilbertian (even if K is not). We can thus apply Case A with Z replaced by $Z \times \mathbb{A}_m$ to finish up the proof in Case B. ∎

Remark 4.3: If G is a compact subgroup of $\mathrm{GL}_n(\mathbb{Q}_\ell)$, then $\Phi(G)$ is open in G (cf. [5, Thm. 8.33], [19, §10.6]). Hence, the above Corollary 4.2 can be applied to ℓ-adic representations of $\pi_1(S[\ell^{-1}])$, e.g., to $\rho_{A\times B,\ell}$.

From now on until the end of this section, let $\mathscr{A}$ and $\mathscr{B}$ be abelian schemes over S. For $T \in \mathrm{Sm}_m(S/Z)$ we denote (as in the previous section) by $A_T/R(T)$ (resp. by $B_T/R(T)$) the generic fiber of $\mathscr{A}_T \to T$ (resp. of $\mathscr{B}_T \to T$). Finally, for $\ell \in \mathbb{L}(S)$ we define

$$\rho_\ell := \rho_{A\times B,\ell}.$$

The following lemma is at the core of the rest of the arguments of this section.

Lemma 4.4: *Let $m \in \{0,1,\dots,d\}$ and $T \in \mathrm{Sm}_m(S/Z)$. Let Δ_1, Δ_2 be subsets of $\mathbb{N}$. If there exists an $R(T)$-homomorphism $f : A_T \to B_T$ (resp. $\overline{R(T)}$-homomorphism $f : A_{T,\overline{R(T)}} \to B_{T,\overline{R(T)}}$) such that*

$$\dim(\ker(f)) \in \Delta_1 \quad \textit{and} \quad \dim(\mathrm{im}(f)) \in \Delta_2$$

and if T is ρ_ℓ-generic, then there exists an $R(S)$-homomorphism $F : A \to B$ (resp. $\overline{R(S)}$-homomorphism $F : A_{\overline{R(S)}} \to B_{\overline{R(S)}}$) such that

$$\dim(\ker(F)) \in \Delta_1 \quad \textit{and} \quad \dim(\mathrm{im}(F)) \in \Delta_2.$$

Proof. Let $f : A_T \to B_T$ be an $R(T)$-homomorphism such that we have $\dim(\ker(f)) \in \Delta_1$ and $\dim(\mathrm{im}(f)) \in \Delta_2$, and assume that T is ρ_ℓ-generic. By Lemma 3.8 the specialization maps

$$\begin{aligned} r &:= r_{\mathscr{A},\mathscr{B},T,S} : \mathrm{Hom}_{R(S)}(A,B) \to \mathrm{Hom}_{R(T)}(A_T, B_T),\\ \overline{r} &:= \overline{r}_{\mathscr{A},\mathscr{B},T,S} : \mathrm{Hom}_{\overline{R(S)}}(A_{\overline{R(S)}}, B_{\overline{R(S)}}) \to \mathrm{Hom}_{\overline{R(T)}}(A_{T,\overline{R(T)}}, B_{T,\overline{R(T)}}), \end{aligned}$$

are injective with finite cokernels $C = \mathrm{coker}(r)$ and $\overline{C} = \mathrm{coker}(\overline{r})$. Let $s = |C|$. Then $s \circ f$ lies in the image of r. Thus there exists an $R(S)$-homomorphism $F : A \to B$ such that $r(F) = s \circ f$. Using Lemma 3.6 once more, we see that

$$\begin{aligned} \dim(\ker(F)) &= \dim(\ker(s \circ f)) = \dim(\ker(f)) \in \Delta_1,\\ \dim(\mathrm{im}(F)) &= \dim(\mathrm{im}(s \circ f)) = \dim(\mathrm{im}(f)) \in \Delta_2. \end{aligned}$$

The proof of the respective case can be carried out in a completely analoguous way using $\overline{r}$ instead of r and taking Remark 3.7 into account. ∎

We are ready for our first local-global statement.

Theorem 4.5: *Let U be a dense open subscheme of S. Let $m \in \{0,1,\dots,d\}$. Assume that K is Hilbertian or that $m \geq 1$. Let Δ_1 and Δ_2 be subsets of $\mathbb{N}$. The following are equivalent:*

(a) *There exists an $R(S)$-homomorphism $F : A \to B$ (resp. $\overline{R(S)}$-homomorphism $F : A_{\overline{R(S)}} \to B_{\overline{R(S)}}$) such that* $\dim(\ker(F)) \in \Delta_1$ *and one has* $\dim(\mathrm{im}(F)) \in \Delta_2$.

(b) *For every $T \in \mathrm{Sm}_m(S/Z)$ with $T \subset U$ there exists a $R(T)$-homomorphism $f : A_T \to B_T$ (resp. $\overline{R(T)}$-homomorphism $f : A_{T,\overline{R(T)}} \to B_{T,\overline{R(T)}}$) such that* $\dim(\ker(f)) \in \Delta_1$ *and one has* $\dim(\mathrm{im}(f)) \in \Delta_2$.

Proof. Assume (a) holds true and let $T \in \mathrm{Sm}_m(S/Z)$. There exists $\ell \in \mathbb{L}(T)$ by Lemma 3.1. The existence of f as in (b) is then immediate from the mere existence of the specialization maps $r_{\mathscr{A},\mathscr{B},T,S}$ and $\overline{r}_{\mathscr{A},\mathscr{B},T,S}$ (cf. definitions of maps (11) and (12)) plus the fact that they "respect dimension data" (cf. Lemma 3.6 and Remark 3.7). This proves (a)$\Rightarrow$(b).

We now prove the implication (b)$\Rightarrow$(a). Assume (b) is true. Choose $\ell \in \mathbb{L}(U)$ (cf. Lemma 3.1). By Corollary 4.2 (applied with $S[\ell^{-1}]$ instead of S and with $U[\ell^{-1}]$ instead of U)) there exists $T \in \mathrm{Sm}_m(S[\ell^{-1}]/Z)$ such that

$$T \subset U[\ell^{-1}]$$

and such that T is ρ_ℓ-generic. By (b) there exists an $R(T)$-homomorphism $f : A_T \to B_T$ (resp. $\overline{R(T)}$-homomorphism $f : A_{T,\overline{R(T)}} \to B_{T,\overline{R(T)}}$) such that $\dim(\ker(f)) \in \Delta_1$ and $\dim(\mathrm{im}(f)) \in \Delta_2$. Now (a) follows by Lemma 4.4. ∎

Proof of Theorem A. Apply Theorem 4.5 with

$$\Delta_1 = \{0\} \quad \text{and} \quad \Delta_2 = \{\dim(B)\}$$

(resp. with $\Delta_1 = \mathbb{N}$ and $\Delta_2 = \{\dim(B)\}$, resp. with $\Delta_1 = \{0, 1, \cdots, \dim(A)-1\}$ and $\Delta_2 = \mathbb{N}$, resp. with $\Delta_1 = \{\kappa\}$ and $\Delta_2 = \mathbb{N}$). ∎

COROLLARY 4.6: *Let U be a dense open subscheme of S. Let $m \in \{0, 1, \cdots, d\}$. Assume that K is Hilbertian or that $m \geq 1$.*

(a) *The following are equivalent:*
  (i) *A is not a simple $R(S)$-variety.*
  (ii) *For every $T \in \mathrm{Sm}_m(S/Z)$ with $T \subset U$ the fiber A_T is not a simple $R(T)$-variety.*

(b) *The following are equivalent:*
  (i) *$A_{\overline{R(S)}}$ is not a simple $\overline{R(S)}$-variety.*
  (ii) *For every $T \in \mathrm{Sm}_m(S/Z)$ with $T \subset U$ the fiber $A_{T,\overline{R(T)}}$ is not a simple $\overline{R(T)}$-variety.*

Proof. Note that A is non-simple if and only if there exists

$$\kappa \in \{1, 2, \ldots, \dim(A)-1\}$$

and a homomorphism $A \to A$ with κ-dimensional kernel. Thus the corollary is a formal consequence of Theorem 4.5. ∎

Theorem 4.7: *Let U be a dense open subscheme of S. Let $m\in\{0,1,\dots,d\}$. Assume that K is Hilbertian or that $m \geq 1$. The following are equivalent:*

(a) *A is a quadratic isogeny twist of B.*

(b) *For every $T \in \mathrm{Sm}_m(S/Z)$ with $T \subset U$ the abelian variety A_T is a quadratic isogeny twist of B_T.*

The implication (a)⇒(b) holds true also in the case where $Z = \mathrm{Spec}(k)$ with a finite field k and $m = 0$.

Proof. We prove the implication (a)⇒(b): Assume that A is a quadratic isogeny twist of B and let $T \in \mathrm{Sm}_m(S/Z)$. Assume that $T \subset U$. Choose $\ell \in \mathbb{L}(T)$ (cf. Lemma 3.1). It follows by Lemma 2.6, that there exists a $\chi \in \mathrm{Hom}(\mathrm{Gal}(R(S)), \{\pm\mathrm{Id}_B\})$ such that

$$\rho_{A,\ell} \cong \chi \otimes \rho_{B,\ell}.$$

For every $\sigma \in \ker(\mathrm{Gal}(R(S)) \to \pi_1(U[\ell^{-1}]))$ we have

$$\rho_{A,\ell}(\sigma) = \mathrm{Id}_{T_\ell(A)} \quad \text{and} \quad \rho_{B,\ell}(\sigma) = \mathrm{Id}_{T_\ell(B)},$$

hence $\chi(\sigma) = +\mathrm{Id}_B$, i.e., χ factors through $\pi_1(U[\ell^{-1}])$. Now we have

$$\rho_{A_T,\ell} \cong u^*\rho_{A,\ell} = u^*(\chi \otimes \rho_{B,\ell}) = u^*\chi \otimes u^*\rho_{B,\ell} \cong \chi_T \otimes \rho_{B_T,\ell}$$

where $u{:}T\to S$ is the embedding and $\chi_T{:}\pi_1(T[\ell^{-1}])\to\{\pm\mathrm{Id}_{B_T}\}$ is a quadratic character. By Lemma 2.6 A_T is a quadratic isogeny twist of B_T.

We now prove the other implication (b)⇒(a): Assume that for every $T \in \mathrm{Sm}_m(S/Z)$ with $T \subset U$ the fiber A_T is a quadratic isogeny twist of B_T. Let $\ell \in \mathbb{L}(U)$ (cf. Lemma 3.1). By Corollary 4.2 there exists $T \in \mathrm{Sm}_m(S/Z)$ with $T \subset U[\ell^{-1}]$ such that T is $\rho_{A\times B,\ell}$-generic. Let

$$\mathscr{K} = \ker(\rho_{A\times B,\ell} : \pi_1(S[\ell^{-1}]) \to GL_{T_\ell(A\times B)}(\mathbb{Z}_\ell))$$

and

$$\mathscr{K}_T = \ker(\rho_{A\times B,\ell} \circ u_*).$$

Then $u_* : \pi_1(T) \to \pi_1(S[\ell^{-1}])$ induces an isomorphism

$$\pi_1(T)/\mathscr{K}_T \to \pi_1(S[\ell^{-1}])/\mathscr{K}.$$

By Lemma 2.6 there exists $\chi_T \in \mathrm{Hom}(\pi_1(T), \{\pm\mathrm{Id}_{B_T}\})$ such that

$$\rho_{A_T,\ell} \cong \chi_T \otimes \rho_{B_T,\ell}.$$

It follows that

$$\chi_T(\sigma) = +\mathrm{Id}_{B_T},$$

for every $\sigma \in \mathscr{K}_T$, because $\sigma \in \ker(\rho_{A_T,\ell}) \cap \ker(\rho_{B_T,\ell})$. Hence χ_T factors through $\pi_1(T)/\mathscr{K}_T$ and thus induces a quadratic character χ of $\pi_1(S[\ell^{-1}])/\mathscr{K}$ and thus of $\pi_1(S[\ell^{-1}])$, such that

$$\rho_{A,\ell} \cong \chi \otimes \rho_{B,\ell}.$$

Thus (a) holds true by Lemma 2.6. ∎

5. Specialization to a finite field

In Theorem 4.5, Corollary 4.6 and Theorem 4.7 the case when k is finite and m=0 has been excluded. This case is more difficult and cannot be handled with Hilbertianity alone. In this section we generalize parts of results 4.5, 4.6, 4.7 (under additional assumptions) to this "critical" case by using recent results of Khare–Larsen [13] and Fité [9].

Definition 5.1: Let A be an abelian variety over a finitely generated field K and let $\ell \neq \mathrm{char}(K)$ be a rational prime. Let $\underline{G}_\ell$ be the connected component of the Zariski closure of $\rho_{A,\ell}(\mathrm{Gal}(K))$. We say that A satisfies condition $MWC(A)$ if for all rational primes $\ell \neq \mathrm{char}(K)$ the action of $\underline{G}_{\ell,\overline{\mathbb{Q}}_\ell}$ on each irreducible factor of the representation $V_\ell(A) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ is minuscule in the sense of Bourbaki [1] cf. [13, p. 1].

Remark 5.2: Let A be an abelian variety over a finitely generated field K satisfying $MWC(A)$. Let B be an abelian variety.

(a) For every finite extension K'/K the abelian variety $A_{K'}$ satisfies $MWC(A_{K'})$ because the connected component of the Zariski closure of $\rho_{A,\ell}(\mathrm{Gal}(K'))$ agrees with the connected component of the Zariski closure of $\rho_{A,\ell}(\mathrm{Gal}(K))$ as $\mathrm{Gal}(K')$ is of finite index in $\mathrm{Gal}(K)$.

(b) If $f : A \to B$ is a surjective K-homomorphism, then B satisfies $MWC(B)$, because $V_\ell(f) : V_\ell(A) \to V_\ell(B)$ is a surjective homomorphism of representations.

(c) If B is an abelian subvariety of A, then there exists surjective homomorphism $A \to B$ by [15, Prop. 12.1] and thus B satisfies $MWC(B)$ by (b).

Remark 5.3: If K is a number field, then $MWC(A)$ holds true for every abelian variety A/K due to a result of Pink [16, Cor. 5.11] which proves Zarhin's *minuscule weights conjecture* [24, Conjecture 0.4] in the number field case. The same holds true if K is a finitely generated field of characteristic zero; using Hilbertianity one can easily reduce to the number field case. According to [24, Conjecture 0.4] every abelian variety over a global field of any characteristic should satisfy $MWC(A)$. Zarhin [24, 4.2.1] proved his conjecture for K a global field of positive characteristic and A/K an ordinary abelian variety. In a recent preprint [4] Cadoret and Tamagawa formulated an anologue of Zarhin's conjecture over finitely generated fields and checked that $MWC(A)$ holds true for another case of A over K finitely generated of positive characteristic, see [4, Cor. 6.3.2.3]. On the other hand, the analogue of the conjecture of Zarhin in positive characteristic seems to fail in general, as indicated by a result of Bültel (cf. [2, Thm. 1.2], [2, Comments and examples, p. 637]) who constructed an abelian variety of dimension 56 over a finitely generated field of positive characteristic with a simple factor of exceptional type G_2 (there are no minuscule weights in this case !) in its ℓ-adic monodromy group.[2]

The following is a global function field analogue of a result of Khare and Larsen [13, Thm. 1]. We include the proof for the reader's convenience.

Proposition 5.4: *Let S be a smooth curve over a finite field k. Let $\mathscr{A}, \mathscr{B}$ be abelian schemes over S with generic fibers A and B respectively. Assume that A satisfies $MWC(A)$ and B satisfies $MWC(B)$. If the set of all closed points $s \in S$ such that there exists a non-zero $\overline{k(s)}$-homomorphism $A_{s,\overline{k(s)}} \to B_{s,\overline{k(s)}}$ has Dirichlet density 1, then there exists a non-zero $\overline{k(S)}$-homomorphism $A_{\overline{k(S)}} \to B_{\overline{k(S)}}$.*

Proof. Let $K = R(S)$. Fix one rational prime $\ell \neq \mathrm{char}(K)$. After replacing S by one of its connected finite étale covers, we can assume that the Zariski closures $\underline{G}_A$ of $\rho_{A,\ell}(\mathrm{Gal}(K))$, $\underline{G}_B$ of $\rho_{B,\ell}(\mathrm{Gal}(K))$ and $\underline{G}$ of $\rho_{A\times B,\ell}(\mathrm{Gal}(K))$ are connected. The action of $\underline{G}_{\overline{\mathbb{Q}}_\ell}$ on $V_A := V_\ell(A)\otimes_{\mathbb{Q}_\ell}\overline{\mathbb{Q}}_\ell$ and on $V_B := V_\ell(B)\otimes_{\mathbb{Q}_\ell}\overline{\mathbb{Q}}_\ell$ is minuscule. By [14, Thm. 1.2] and [18] the set of all closed points $s \in S$ such that the Frobenius element $Fr_s \in \underline{G}(\mathbb{Q}_\ell)$ generates a Zariski dense subgroup $\langle Fr_s\rangle$ of a maximal torus of $\underline{G}$ has positive density. Thus we can choose a closed

[2] For more examples of abelian varieties of this type the reader can consult [3] which appeared in the arXiv after our paper had been submitted for publication.

point $s \in S$ such that $\langle Fr_s \rangle$ is a Zariski dense subgroup of a maximal torus $\underline{\mathbb{T}}$ of $\underline{G}$ and such that there exists a non-zero homomorphism $f : A_{s,\overline{k(s)}} \to B_{s,\overline{k(s)}}$. Furthermore, f is defined over a finite extension k' of $k(s)$. Now $|k'| = |k(s)|^m$, for a natural number m and thus f is fixed under the action of Fr_s^m, and Fr_s^m also generates a Zariski dense subgroup of $\underline{\mathbb{T}}$. It follows that the map

$$V_\ell(f) \otimes \overline{\mathbb{Q}}_\ell : V_A \to V_B$$

induced by f is fixed by the torus $\underline{\mathbb{T}}$. Moreover, $V_\ell(f) \otimes \overline{\mathbb{Q}}_\ell$ is a non-zero element of $\mathrm{Hom}_{\underline{\mathbb{T}}}(V_A, V_B)$. Thus

$$\dim(\mathrm{Hom}_{\underline{\mathbb{T}}}(V_A, V_B)) > 0.$$

Now $\underline{G}$ is a reductive group because $\rho_{A\times B,\ell}$ is semisimple. By our assumption A satisfies $MWC(A)$ and B satisfies $MWC(B)$. Thus we can apply [13, Prop. 2] to conclude that

$$\dim(\mathrm{Hom}_{\underline{G}}(V_A, V_B)) > 0.$$

From $\mathrm{Hom}_{\underline{G}}(V_A, V_B) \cong \mathrm{Hom}_K(A, B) \otimes \overline{\mathbb{Q}}_\ell$ and the fact that $\mathrm{Hom}_K(A, B)$ is $\mathbb{Z}$-free it follows that $\mathrm{Hom}_K(A, B)$ is non-zero, as desired. ∎

We can now treat higher dimensional S by combining Proposition 5.4 with the results of the previous section.

Theorem 5.5: *Let S be a smooth variety over a finite field k. Let $\mathscr{A}, \mathscr{B}$ be abelian schemes over S with generic fibers A and B respectively. Assume that A satisfies $MWC(A)$ and B satisfies $MWC(B)$. The following are equivalent:*

(a) *There exists a non-zero $\overline{k(S)}$-homomorphism $A_{\overline{k(S)}} \to B_{\overline{k(S)}}$.*

(b) *For every closed point $s \in S$ there exists a non-zero $\overline{k(s)}$-homomorphism $A_{s,\overline{k(s)}} \to B_{s,\overline{k(s)}}$.*

Proof. After replacing S by one of its connected finite étale covers we can assume that

$$\mathrm{Hom}_{k(S)}(A, B) = \mathrm{Hom}_{\overline{k(S)}}(A_{\overline{k(S)}}, B_{\overline{k(S)}}).$$

The implication (a)⇒(b) is immediate from Lemma 3.6 and Remark 3.7.

We prove the other implication (b)⇒(a): Let $K = R(S)$ and let $\ell \neq \mathrm{char}(K)$ be a rational prime. By Corollary 4.2 there exists a $T \in \mathrm{Sm}_1(S/k)$ of S such that T is $\rho_{A\times B,\ell}$-generic. Then T is automatically $\rho_{A,\ell}$-generic and $\rho_{B,\ell}$-generic. In particular,

$$\rho_{A,\ell}(\mathrm{Gal}(K))^{\mathrm{Zar}} = \rho_{A_T,\ell}(\mathrm{Gal}(k(T)))^{\mathrm{Zar}}.$$

In particular, $A_T/k(T)$ satisfies $MWC(A_T)$. Similarly $B_T/k(T)$ satisfies $MWC(B_T)$. By (b), for every closed point $t \in T$ there exists a non-zero homomorphism $A_{\overline{k(t)}} \to B_{\overline{k(t)}}$. By Proposition 5.4 there exists a non-zero homomorphism $A_{T,\overline{k(T)}} \to B_{T,\overline{k(T)}}$. Now (a) follows by Lemma 4.4. ∎

We can upgrade Theorem 5.5 a bit, to treat not only non-zero homomorphisms but also other important classes of homomorphisms, much in the spirit of Corollary B.

THEOREM 5.6: *Let S be a smooth variety over a finite field k. Let $\mathscr{A}, \mathscr{B}$ be abelian schemes over S with generic fibers A and B respectively. Assume that A satisfies $MWC(A)$ and B satisfies $MWC(B)$. The following are equivalent:*

- (a) *There exists a surjective $\overline{k(S)}$-homomorphism (resp. $\overline{k(S)}$-isogeny) $A_{\overline{k(S)}} \to B_{\overline{k(S)}}$.*
- (b) *For every closed point $s \in S$ there exists a surjective $\overline{k(s)}$-homomorphism (resp. $\overline{k(s)}$-isogeny) $A_{s,\overline{k(s)}} \to B_{s,\overline{k(s)}}$.*

Proof. The implication (a)⇒(b) is immediate from Lemma 3.6.

We prove the implication (b)⇒ (a) in case "surjective". After replacing S by one of its connected finite étale covers we can assume that

$$\begin{aligned} \operatorname{Hom}_{k(S)}(A,B) &= \operatorname{Hom}_{\overline{k(S)}}(A_{\overline{k(S)}}, B_{\overline{k(S)}}),\\ \operatorname{End}_{k(S)}(A) &= \operatorname{End}_{\overline{k(S)}}(A_{\overline{k(S)}}),\\ \operatorname{End}_{k(S)}(B) &= \operatorname{End}_{\overline{k(S)}}(B_{\overline{k(S)}}). \end{aligned} \tag{13}$$

By the Poincaré reducibility theorem (cf. [15, Prop. 12.1] and the passage below) A (resp. B) is $k(S)$-isogenous to $\prod_{i\in I} A_i^{n_i}$ (resp. $\prod_{j\in J} B_j^{m_i}$) where I and J are finite sets, the A_i $(i \in I)$ are mutually not $k(S)$-isogenous simple abelian varieties over $k(S)$ and the B_j $(j \in J))$ are mutually not $k(S)$-isogenous simple abelian varieties over $k(S)$. We consider the commutative diagram

$$\begin{array}{ccc} \operatorname{End}^0_{\overline{k(S)}}(A_{\overline{k(S)}}) & \longrightarrow & \prod_{i,j\in I} \operatorname{Hom}^0_{\overline{k(S)}}(A_{i,\overline{k(S)}}, A_{j,\overline{k(S)}})^{n_i\times n_j} \\ \uparrow & & \uparrow {\scriptstyle \prod_{i,j\in I} s_{ij}^{n_i\times n_j}} \\ \operatorname{End}^0_{k(S)}(A_{k(S)}) & \longrightarrow & \prod_{i,j\in I} \operatorname{Hom}^0_{k(S)}(A_i, A_j)^{n_i\times n_j} \end{array}$$

where the

$$s_{ij} : \operatorname{Hom}^0_{k(S)}(A_i, A_j) \to \operatorname{Hom}^0_{\overline{k(S)}}(A_{i,\overline{k(S)}}, A_{j,\overline{k(S)}})$$

are the canonical maps. The horizontal maps in the diagram are bijective. The left hand vertical map is bijective by (13). Thus the s_{ij} are bijective, too. For $i \neq j$ we have $A_i \not\sim A_j$, hence

$$\mathrm{Hom}^0_{\overline{k(S)}}(A_{i,\overline{k(S)}}, A_{j,\overline{k(S)}}) = \mathrm{Hom}^0_{k(S)}(A_i, A_j) = 0,$$

and thus the $A_{i,\overline{k(S)}}$ are mutually non-isogenous over $\overline{k(S)}$. As A_i is simple it follows that $\mathrm{End}^0_{\overline{k(S)}}(A_{i,\overline{k(S)}}) = \mathrm{End}^0_{k(S)}(A_i)$ is a division algebra over $\mathbb{Q}$ and thus the $A_{i,\overline{k(S)}}$ are simple. Likewise the $B_{j,\overline{k(S)}}$ are simple and mutually non-isogenous over $\overline{k(S)}$. Note that $MWC(A_i)$ and $MWC(B_j)$ hold true (cf. Remark 5.2).

After replacing S by one of its dense open subschemes each A_i (resp. B_j) extends to an abelian scheme $\mathscr{A}_i$ (resp. $\mathscr{B}_j$) over S. It is then clear that for every closed point $s \in S$ the fiber $A_{s,\overline{k(s)}}$ is $\overline{k(s)}$-isogenous to $\prod_{i\in I} A^{n_i}_{i,s,\overline{k(s)}}$ and $B_{s,\overline{k(s)}}$ is $\overline{k(s)}$-isogenous to $\prod_{j\in J} B^{m_j}_{j,s,\overline{k(s)}}$. In what follows we can thus asssume, without loss of generality, that $A = \prod_{i\in I} A_i^{n_i}$ and that $B = \prod_{j\in J} B_j^{m_j}$. From (b) we know that for every closed point $s \in S$ there exists a surjective $\overline{k(s)}$-homomorphism $A_{s,\overline{k(s)}} \to B_{s,\overline{k(s)}}$ and thus for every $j \in J$ a surjective homomorphism $A_{s,\overline{k(s)}} \to B_{j,s,\overline{k(s)}}$.

By Theorem 5.5 there exists for every $j \in J$ a non-zero homomorphism $A_{\overline{k(S)}} \to B_{j,\overline{k(S)}}$. Thus, for every $j \in J$, there must be a unique $\nu(j) \in I$, such that there exists a non-zero $\overline{k(S)}$-homomorphism $h_j : A_{\nu(j),\overline{k(S)}} \to B_{j,\overline{k(S)}}$ and h_j must be a $\overline{k(S)}$-isogeny, because $A_{\nu(j),\overline{k(S)}}$ and $B_{j,\overline{k(S)}}$ are both simple.

We now show that $n_{\nu(j)} \geq m_j$, for all $j \in J$. Let $j \in J$ be arbitrary. Let

$$A'_j = \prod_{i\in I\setminus\{\nu(j)\}} A_i^{n_i}.$$

Then for every $j \in J$ we have

$$A = A^{n_{\nu(j)}}_{\nu(j)} \times A'_j.$$

There is no non-zero $\overline{k(S)}$-homomorphism $A'_{j,\overline{k(S)}} \to B_{j,\overline{k(S)}}$, because the $A_{i,\overline{k(S)}}$ are mutually non-isogenous over $\overline{k(S)}$ and $A_{\nu(j),\overline{k(S)}} \simeq B_{j,\overline{k(S)}}$. The contraposition of Theorem 5.5 implies that there exists a closed point $s \in S$ such that there is no non-zero $\overline{k(s)}$-homomorphism $A'_{j,s,\overline{k(s)}} \to B_{j,s,\overline{k(s)}}$. Furthermore, by (b), there is a surjective $\overline{k(s)}$-homomorphism

$$A_{s,\overline{k(s)}} = A^{n_{\nu(j)}}_{\nu(j),s,\overline{k(s)}} \times A'_{j,s,\overline{k(s)}} \to B^{m_j}_{j,s,\overline{k(s)}}$$

which restricts to a surjective $\overline{k(s)}$-homomorphism

$$A^{n_{\nu(j)}}_{\nu(j),s,\overline{k(s)}} \times \{0\} \to B^{m_j}_{j,s,\overline{k(s)}}.$$

As $\dim(A_{\nu(j),s,\overline{k(s)}}) = \dim(B_{j,s,\overline{k(s)}})$, this finally shows that $n_{\nu(j)} \geq m_j$ as claimed above.

This together with the existence of the isogeny h_j implies the existence of a surjective $\overline{k(S)}$-homomorphism $f_j : A^{n_{\nu(j)}}_{\nu(j),\overline{k(S)}} \to B^{m_j}_{j,\overline{k(S)}}$, and these in turn induce a surjective $\overline{k(S)}$-homomorphism

$$\prod_{j\in J} f_j : \prod_{j\in J} A^{n_{\nu(j)}}_{\nu(j),\overline{k(S)}} \to \prod_{j\in J} B^{m_j}_{j,\overline{k(S)}} = B_{\overline{k(S)}}. \tag{14}$$

Composing the surjective $\overline{k(S)}$-homomorphism (14) with the projection

$$A_{\overline{k(S)}} \to \prod_{j\in J} A^{n_{\nu(j)}}_{\nu(j),\overline{k(S)}}$$

we obtain a $\overline{k(S)}$-epimorphism $A_{\overline{k(S)}} \to B_{\overline{k(S)}}$ as desired.

We next prove the implication (b)$\Rightarrow$(a) in case "isogeny". In that case we know by (b) that for every closed point $s \in S$ there exists a $\overline{k(s)}$-isogeny

$$g_s : A_{s,\overline{k(s)}} \to B_{s,\overline{k(s)}}.$$

The above arguments imply the existence of a surjective $\overline{k(S)}$-homomorphism $g : A_{\overline{k(S)}} \to B_{\overline{k(S)}}$. Let $s \in S$ be a closed point. Then, by the existence of the isogeny g_s,

$$\dim(A) = \dim(A_{s,\overline{k(s)}}) = \dim(B_{s,\overline{k(s)}}) = \dim(B),$$

and thus g must automatically be an isogeny. ∎

We shall now establish a global function field analogue of [9, Cor. 2.7], following proof in loc. cit. quite closely. For this we need the following theorem of Rajan [17]. Let K be a global field and denote by Σ_K the set of all non-archimedian discrete valuations of K. Let $\ell \neq \mathrm{char}(K)$ be a rational prime and let $\rho_1 : \mathrm{Gal}(K) \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ and $\rho_2 : \mathrm{Gal}(K) \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ be continuous semisimple representations which are unramified outside a finite set $S \subset \Sigma_K$. Define

$$SM(\rho_1, \rho_2) := \{v \in \Sigma_K : Tr(\rho_1(Fr_v)) = Tr(\rho_2(Fr_v))\}.$$

Theorem 5.7 (Rajan, [17, Thm. 2]): *Suppose that the Zariski closure H_1 of $\rho_1(\mathrm{Gal}(K))$ is connected and that the upper Dirichlet density of $SM(\rho_1, \rho_2)$ is positive. Then the following hold true:*

(a) *There is a finite Galois extension L/K such that $\rho_1|_{\mathrm{Gal}(L)} \cong \rho_2|_{\mathrm{Gal}(L)}$ and the connected component of the Zariski closure of $\rho_2(\mathrm{Gal}(K))$ is conjugate to H_1.*

(b) *If ρ_1 is absolutely irreducible, then there is a Dirichlet character $\chi : \mathrm{Gal}(L/K) \to \mathbb{Q}_\ell^\times$ of finite order such that*

$$\rho_2 \cong \chi \otimes_{\mathbb{Q}_\ell} \rho_1.$$

We have the following global function field analogue of [9, Cor. 2.7].

Proposition 5.8: *Let S be a smooth curve over a finite field k. Let $\mathscr{A}, \mathscr{B}$ be abelian schemes over S with generic fibers A and B, respectively. Let $\ell \neq \mathrm{char}(k)$ be a rational prime and let $K = k(S)$. Let U be a dense open subscheme of S. Assume that $\mathrm{End}_{\overline{k(S)}}(A) = \mathrm{End}_{\overline{k(S)}}(B) = \mathbb{Z}$ and that the Zariski closures of $\rho_{A,\ell}(\mathrm{Gal}(K))$ and of $\rho_{B,\ell}(\mathrm{Gal}(K))$ are connected. If the set of all closed points $u \in U$ such that A_u is a quadratic isogeny twist of B_u has Dirichlet density 1, then A is a quadratic isogeny twist of B.*

Proof. Let Γ be the set of all $u \in U$ such that A_u is a quadratic isogeny twist of B_u. Assume that Γ has Dirichlet density 1. We claim that $SM(\rho_{\ell,A}, \rho_{\ell,B})$ has positive upper Dirichlet density. Consider the two sets

$$\Gamma_\pm := \{u \in \Gamma : Tr(\rho_{A,\ell}(Fr_u)) = \pm Tr(\rho_{B,\ell}(Fr_u))\}.$$

For every $u \in \Gamma$ we have $Tr(\rho_{A,\ell}(Fr_u)) \in \{\pm Tr(\rho_{B,\ell}(Fr_u))\}$ by Lemma 2.6 (implication (a)$\Rightarrow$(c) applied over $k(u)$). Thus $\Gamma = \Gamma_+ \cup \Gamma_-$. Moreover

$$\Gamma_+ \subset SM(\rho_{\ell,A}, \rho_{\ell,B}).$$

Because otherwise it would follow that Γ_+ has upper Dirichlet density zero and that Γ_- has Dirichlet density 1. But then the Chebotarev density theorem would imply that $Tr(\rho_{A,\ell}(g)) = -Tr(\rho_{B,\ell}(g))$ for all $g \in \mathrm{Gal}(K)$, which is obviously false for $g = \mathrm{Id}$. Thus the inclusion $\Gamma_+ \subset SM(\rho_{\ell,A}, \rho_{\ell,B})$ holds true.

Our assumption

$$\mathrm{End}_{\overline{k(S)}}(A) = \mathrm{End}_{\overline{k(S)}}(B) = \mathbb{Z}$$

implies that $\rho_{A,\ell}$ and $\rho_{B,\ell}$ are absolutely irreducible.

Now, because of the claim, Theorem 5.7 implies that there is a finite Galois extension L/K and a character $\chi : \mathrm{Gal}(L/K) \to \mathbb{Q}_\ell^\times$ of finite order such that $\rho_{A,\ell} = \chi \otimes \rho_{B,\ell}$. But now, by Theorem 2.3, we have an isomorphism of $\mathrm{Gal}(L/K)$-modules

$$\begin{aligned}\mathrm{Hom}_L(A_L, B_L) \otimes \mathbb{Q}_\ell &\cong V_\ell(A)^* \otimes_{\mathbb{Q}_\ell} V_\ell(B) \\ &\cong V_\ell(A)^* \otimes_{\mathbb{Q}_\ell} V_\ell(A) \otimes \mathbb{Q}_\ell(\chi) \\ &\cong \mathrm{End}_L(A_L) \otimes \mathbb{Q}_\ell(\chi) \cong \mathbb{Q}_\ell(\chi).\end{aligned}$$

Thus $\mathrm{Gal}(L/K)$ acts on $\mathrm{Hom}_L(A_L, B_L) \cong \mathbb{Z}$ via χ and this implies that χ is a quadratic character. Lemma 2.6 (implication (b)$\Rightarrow$ (a) applied over K) now implies that B is a quadratic isogeny twist of A. ∎

Again one can eliminate the assumption $\dim(S) = 1$ by the Hilbertianity approach of the previous section.

Corollary 5.9: *Let S be a smooth variety over a finite field k. Let $\mathscr{A}, \mathscr{B}$ be abelian schemes over S with generic fibers A and B respectively. Let U be a dense open subscheme of S. Assume that $\mathrm{End}_{\overline{k(S)}}(A) = \mathrm{End}_{\overline{k(S)}}(B) = \mathbb{Z}$. The following are equivalent:*

- (a) *A is a quadratic isogeny twist of B.*
- (b) *For every closed point u of U the abelian variety A_u is a quadratic isogeny twist of B_u.*

Proof. The implication (a)$\Rightarrow$(b) is known from Theorem 4.7. We prove the other implication. Assume that for every closed point u of U the abelian variety A_u is a quadratic isogeny twist of B_u. Then, for every smooth connected curve T on U the abelian variety A_T is a quadratic isogeny twist of B_T by Proposition 5.8. Thus Theorem 4.7 implies (a). ∎

Remark 5.10: It is easy to show that if A and B in Corollary 5.9 are elliptic curves with nontrivial endomorphisms, then the claim still holds. It is so, since then j-invariants $j(A)$ and $j(B)$ belong to $\overline{k}$, hence the curves are isotrivial, cf. [21, V.3.1 and Ex. V.5.8], so a twist between fibers at $u \in U$ extends to a twist between the curves. It is an interesting question, if Corollary 5.9 holds true for nonsimple abelian varieties, e.g., for products of mutually nonisogenous elliptic curves over $k(S)$.

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