

# Big monodromy theorem for abelian varieties over finitely generated fields

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## Abstract

An abelian variety over a field  $K$  is said to have *big monodromy*, if the image of the Galois representation on  $\ell$ -torsion points, for almost all primes  $\ell$ , contains the full symplectic group. We prove that all abelian varieties over a finitely generated field  $K$  with the endomorphism ring  $\mathbb{Z}$  and semistable reduction of toric dimension one at a place of the base field  $K$  have big monodromy. We make no assumption on the transcendence degree or on the characteristic of  $K$ . This generalizes a recent result of Chris Hall.

## Introduction

It has been known in number theory, since times immemorial that Galois representation attached to the action of the absolute Galois group on torsion points

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of an abelian group scheme carries a lot of basic arithmetic and geometric information. The first aim which one encounters naturally, while studying such representations is to determine their images in terms of linear algebraic groups. There exists a vast variety of results in the literature concerning computations of Galois representations for abelian varieties defined over number fields and their applications to some classical questions such as Hodge, Tate and Mumford-Tate conjectures, see for example [30], [?], [?], [?] or [?]. In this paper we are interested in computing images of Galois representations attached to abelian varieties defined over finitely generated fields in arbitrary characteristic, i.e., to families of abelian varieties.

Let  $K$  be a field and denote by  $G_K$  its absolute Galois group. Let  $A/K$  be an abelian variety and  $\ell \neq \text{char}(K)$  a prime number. We denote by  $\rho_{A[\ell]} : G_K \rightarrow \text{Aut}(A[\ell])$  the Galois representation attached to the action of  $G_K$  on the  $\ell$ -torsion points of  $A$ . We define  $\mathcal{M}_K(A[\ell]) := \rho_{A[\ell]}(G_K)$  and call this group *the mod- $\ell$  monodromy group of  $A/K$* . We fix a polarization and denote by  $e_\ell : A[\ell] \times A[\ell] \rightarrow \mu_\ell$  the corresponding Weil pairing. Then  $\mathcal{M}_K(A[\ell])$  is a subgroup of the group of symplectic similitudes  $\text{GSp}(A[\ell], e_\ell)$  of the Weil pairing. We will say that  $A/K$  *has big monodromy* if there exists a constant  $\ell_0$  such that  $\mathcal{M}_K(A[\ell])$  contains the symplectic group  $\text{Sp}(A[\ell], e_\ell)$ , for every prime number  $\ell \geq \ell_0$ . Note that the property of having big monodromy does not depend on the choice of the polarization.

Certainly, the most prominent result on computing monodromy groups is the classical theorem of Serre (cf. [27], [28]): *If  $A$  is an abelian variety over a finitely generated field  $K$  of characteristic zero with  $\text{End}(A) = \mathbb{Z}$  and  $\dim(A) = 2, 6$  or odd, then  $A/K$  has big monodromy.* In this paper we consider monodromies for abelian varieties over finitely generated fields which have been recently investigated by Chris Hall [14], [15]. To simplify notation, we will say that an abelian variety  $A$  over a finitely generated field  $K$  *is of Hall type*, if  $\text{End}(A) = \mathbb{Z}$  and  $K$  has a discrete valuation at which  $A$  has semistable reduction of toric dimension one.

In the special case, when  $K = F(t)$  is a rational function field over another finitely generated field, it has been shown by Hall that certain hyperelliptic Jacobians have big monodromy; namely the Jacobians  $J_C$  of hyperelliptic curves  $C/K$  with affine equation  $C : Y^2 = (X - t)f(X)$ , where  $f \in F[X]$  is a monic squarefree polynomial of even degree  $\geq 4$  (cf. [14, Theorem 5.1]). Furthermore, Hall has proved recently [15] the following theorem which in our notation reads: *If  $K$  is a global field, then every abelian variety  $A/K$  of Hall type has big monodromy.* We strengthen these results in our main theorem as follows.

**Main Theorem.** *[cf. Thm. 3.6] If  $K$  is a finitely generated field (of arbitrary characteristic) and  $A/K$  is an abelian variety of Hall type, then  $A/K$  has big monodromy.*

Our proof of the main Theorem follows Hall's proof of [15] to some extent, e.g., we have borrowed a group theory result from [15] (cf. Theorem 3.4). In addition to that we had to apply a substantial quantity of new methods to achieve the extension to all finitely generated fields, such as for instance finite

generation properties of fundamental groups of schemes and Galois theory of certain division fields of abelian varieties, which are gathered in Section 2 and Section 3 of the paper. Furthermore, at a technical point in the case  $\text{char}(K) = 0$ , we perform a tricky reduction argument (described in detail in Section 3) at a place of  $K$  whose residue field is a number field. The paper carries an appendix with a self contained proof of the group theoretical Theorem 3.4 due to Hall, which can be of independent value for the reader.

Theorem A plays an important role in our paper [?], where we make progress on the conjecture of Geyer and Jarden (cf. [9]) on torsion of abelian varieties over large algebraic extensions of finitely generated fields.

As a further application, we combine our monodromy computation with recent results of Ellenberg, Hall and Kowalski in order to obtain the following result on endomorphism rings and simplicity of fibres in certain families of abelian varieties. If  $K$  is a finitely generated transcendental extension of another field  $F$  and  $A/K$  is an abelian variety, then we call  $A$  *weakly isotrivial with respect to  $F$* , if there is an abelian variety  $B/\tilde{F}$  and an  $\tilde{K}$ -isogeny  $B_{\tilde{K}} \rightarrow A_{\tilde{K}}$ .

**Corollary.** *[cf. Cor. 4.3] Let  $F$  be a finitely generated field and  $K = F(t)$  the function field of  $\mathbb{P}^1/F$ . Let  $A/K$  be an abelian variety. Let  $U \subset \mathbb{P}^1$  be an open subscheme such that  $A$  extends to an abelian scheme  $\mathcal{A}/U$ . For  $u \in U(F)$  denote by  $A_u/F$  the corresponding special fiber of  $\mathcal{A}$ . Assume that  $A$  is not weakly isotrivial with respect to  $F$  and that either of the conditions i) or ii) listed below is satisfied.*

i)  $A$  is of Hall type.

ii)  $\text{char}(K) = 0$ ,  $\text{End}(A) = \mathbb{Z}$  and  $\dim(A) = 2, 6$  or odd.

Then the sets:

$$X_1 := \{u \in U(F) \mid \text{End}(A_u) \neq \mathbb{Z}\}$$

and

$$X_2 := \{u \in U(F) \mid A_u/F \text{ is not geometrically simple}\}$$

are finite.

Note that Ellenberg, Elsholtz, Hall and Kowalski proved the statement of the Corollary in the special case when  $A$  is the Jacobian variety of the hyperelliptic curve given by the affine equation  $Y^2 = (X - t)f(X)$ , with  $f \in F[X]$  squarefree and monic of even degree  $\geq 4$  (cf. [5, Theorem 8]). It is the case, where the monodromy of  $A$  is known by [14, Theorem 5.1]. We obtain part (i) of the Corollary as a consequence of the main theorem, our Proposition 4.2 below and also Propositions 4 and 7 of [5]. In order to prove (ii) we use Serre's Theorem [27], [28] instead of the main Theorem.

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# 1 Notation and background material

In this section we fix notation and gather some background material on Galois representations that is important for the rest of this paper.

Let  $X$  be a scheme. For  $x \in X$  we denote by  $k(x)$  the residue field at  $x$ . If  $X$  is integral, then  $R(X)$  stands for the function field of  $X$ , that is, for the residue field at the generic point of  $X$ . If  $X$  happens to be a scheme of finite type over a base field  $F$ , then we often write  $F(x)$  instead of  $k(x)$  and  $F(X)$  instead of  $R(X)$ .

If  $K$  is a field, then we denote by  $K_{\text{sep}}$  (resp.  $\tilde{K}$ ) the separable (resp. algebraic) closure of  $K$  and by  $G_K$  its absolute Galois group. A finitely generated field is by definition a field which is finitely generated over its prime field. For an abelian variety  $A/K$  we let  $\text{End}_K(A)$  be the ring of all  $K$ -endomorphisms of  $A$ . We denote by  $\text{End}(A) := \text{End}_{\tilde{K}}(A_{\tilde{K}})$  the *absolute* endomorphism ring.

If  $\Gamma$  is an object in an abelian category and  $n \in \mathbb{Z}$ , then  $n_\Gamma : \Gamma \rightarrow \Gamma$  is the morphism “multiplication by  $n$ ” and  $\Gamma[n]$  is the kernel of  $n_\Gamma$ . Recall that there is an equivalence of categories between the category of finite étale group schemes over  $K$  and the category of finite (discrete)  $G_K$ -modules, where we attach  $\Gamma(K_{\text{sep}})$  to a finite étale group scheme  $\Gamma/K$ . For such a finite étale group scheme  $\Gamma/K$  we sometimes write just  $\Gamma$  instead of  $\Gamma(K_{\text{sep}})$ , at least in situations where we are sure that this does not cause any confusion. For example, if  $A/K$  is an abelian variety and  $n$  an integer coprime to  $\text{char}(K)$ , then we often write  $A[n]$  rather than  $A(K_{\text{sep}})[n]$ . Furthermore we put  $A[n^\infty] := \bigcup_{i \in \mathbb{N}} A[n^i]$ .

If  $M$  is a  $G_K$ -module (for example  $M = \mu_n$  or  $M = A[n]$  where  $A/K$  is an abelian variety), then we shall denote the corresponding representation of the Galois group  $G_K$  by

$$\rho_M : G_K \rightarrow \text{Aut}(M)$$

and define  $\mathcal{M}_K(M) := \rho_M(G_K)$ . We define  $K(M) := K_{\text{sep}}^{\ker(\rho_M)}$  to be the fixed field in  $K_{\text{sep}}$  of the kernel of  $\rho_M$ . Then  $K(M)/K$  is a Galois extension and  $G(K(M)/K) \cong \mathcal{M}_K(M)$ .

If  $R$  is a commutative ring with 1 (usually  $R = \mathbb{F}_\ell$  or  $R = \mathbb{Z}_\ell$ ) and  $M$  is a finitely generated free  $R$ -module equipped with a non-degenerate alternating bilinear pairing  $e : M \times M \rightarrow R'$  into a free  $R'$ -module of rank 1 (which is a multiplicatively written  $R$ -module in our setting below), then we denote by

$$\text{Sp}(M, e) = \{f \in \text{Aut}_R(M) \mid \forall x, y \in M : e(f(x), f(y)) = e(x, y)\}$$

the corresponding symplectic group and by

$$\text{GSp}(M, e) = \{f \in \text{Aut}_R(M) \mid \exists \varepsilon \in R^\times : \forall x, y \in M : e(f(x), f(y)) = \varepsilon e(x, y)\}$$

the corresponding group of symplectic similitudes.

Let  $n$  be an integer coprime to  $\text{char}(K)$  and  $\ell$  be a prime different from  $\text{char}(K)$ . Let  $A/K$  be an abelian variety. We denote by  $A^\vee$  the dual abelian variety and

let  $e_n : A[n] \times A^\vee[n] \rightarrow \mu_n$  and  $e_{\ell^\infty} : T_\ell A \times T_\ell A^\vee \rightarrow \mathbb{Z}_\ell(1)$  be the corresponding Weil pairings. If  $\lambda : A \rightarrow A^\vee$  is a polarization, then we deduce Weil pairings  $e_n^\lambda : A[n] \times A[n] \rightarrow \mu_n$  and  $e_{\ell^\infty}^\lambda : T_\ell A \times T_\ell A \rightarrow \mathbb{Z}_\ell(1)$  in the obvious way. If  $\ell$  does not divide  $\deg(\lambda)$  and if  $n$  is coprime to  $\deg(\lambda)$ , then  $e_n^\lambda$  and  $e_{\ell^\infty}^\lambda$  are non-degenerate, alternating,  $G_K$ -equivariant pairings. Hence we have representations

$$\begin{aligned} \rho_{A[n]} : G_K &\rightarrow \mathrm{GSp}(A[n], e_n^\lambda), \\ \rho_{T_\ell A} : G_K &\rightarrow \mathrm{GSp}(T_\ell A, e_{\ell^\infty}^\lambda) \end{aligned}$$

with images  $\mathcal{M}_K(A[n]) \subset \mathrm{GSp}(A[n], e_n^\lambda)$  and  $\mathcal{M}_K(T_\ell A) \subset \mathrm{GSp}(T_\ell A, e_{\ell^\infty}^\lambda)$ . We shall say that an abelian variety  $(A, \lambda)$  over a field  $K$  has *big monodromy*, if there is a constant  $\ell_0 > \max(\mathrm{char}(K), \deg(\lambda))$  such that  $\mathcal{M}_K(A[\ell]) \supset \mathrm{Sp}(A[\ell], e_\ell^\lambda)$  for every prime number  $\ell \geq \ell_0$ .

Now let  $S$  be a noetherian regular 1-dimensional connected scheme with function field  $K = R(S)$  and  $A/K$  an abelian variety. Denote by  $\mathcal{A} \rightarrow S$  the Néron model (cf. [2]) of  $A$ . For  $s \in S$  let  $A_s := \mathcal{A} \times_S \mathrm{Spec}(k(s))$  be the corresponding fiber. Recall that we say that  $A$  has *good reduction at  $s$*  provided  $A_s$  is an abelian variety. In general, we denote by  $A_s^\circ$  the connected component of  $A_s$ . If  $T$  is a maximal torus in  $A_s^\circ$ , then  $\dim(T)$  does not depend on the choice of  $T$  [13, IX.2.1] and we call  $\dim(T)$  the *toric dimension* of the reduction  $A_s$  of  $A$  at  $s$ . Finally recall that one says that  $A$  has *semi-stable reduction at  $s$* , if  $A_s^\circ$  is an extension of an abelian variety by a torus.

We shall also need the following connections between the reduction type of  $A$  and properties of the Galois representations attached to  $A$ . Let  $s$  be a closed point of  $S$ . The valuation  $v$  attached to  $s$  admits an extension to the separable closure  $K_{\mathrm{sep}}$ ; we choose such an extension  $\bar{v}$  and denote by  $D(\bar{v})$  the corresponding decomposition group. This is the absolute Galois group of the quotient field  $K_s = Q(\mathcal{O}_{S,s}^h)$  of the henselization  $\mathcal{O}_{S,s}^h$  of the valuation ring  $\mathcal{O}_{S,s}$  of  $v$ . Hence the results mentioned in [13, I.0.3] for the henselian case carry over to give the following description of  $D(\bar{v})$ : If  $I(\bar{v})$  is the kernel of the canonical map  $D(\bar{v}) \rightarrow G_{k(s)}$  defined by  $\bar{v}$ , then  $D(\bar{v})/I(\bar{v}) \cong G_{k(s)}$ . Let  $p$  be the characteristic of the residue field  $k(s)$  ( $p$  is zero or a prime number).  $I(\bar{v})$  has a maximal pro- $p$  subgroup  $P(\bar{v})$  ( $P(\bar{v}) = 0$  if  $p = 0$ ) and

$$I(\bar{v})/P(\bar{v}) \cong \varprojlim_{n \notin p\mathbb{Z}} \mu_n(k(s)_{\mathrm{sep}}) \cong \prod_{\ell \neq p \text{ prime}} \mathbb{Z}_\ell(1).$$

Hence the maximal pro- $\ell$ -quotient  $I_\ell(\bar{v})$  of  $I(\bar{v})$  is isomorphic to  $\mathbb{Z}_\ell(1)$ , if  $\ell \neq p$  is a prime.

**Proposition 1.1.** *Let  $\ell \neq p$  be a prime number. Assume that  $A$  has semi-stable reduction at  $s$ .*

- a) *The image  $\rho_{A[\ell]}(P(\bar{v})) = \{Id\}$  and  $\rho_{A[\ell]}(I(\bar{v}))$  is a cyclic  $\ell$ -group.*
- b) *Let  $g$  be a generator of  $\rho_{A[\ell]}(I(\bar{v}))$ . Then  $(g - Id)^2 = 0$ .*
- c) *Assume that  $\ell$  does not divide the order of the component group of  $A_s$ . The toric dimension of  $A$  at  $s$  is equal to  $2 \dim(A) - \dim_{\mathbb{F}_\ell}(\mathrm{Eig}(g, 1))$  if  $\mathrm{Eig}(g, 1) = \ker(g - Id)$  is the eigenspace of  $g$  at 1.*

*Proof.* Part a) and b) are immediate consequences of [13, IX.3.5.2].

Assume from now on that  $\ell$  does not divide the order of the component group of  $A_s$ . This assumption implies  $A_s^\circ[\ell] \cong A_s[\ell]$ .

As we assumed  $A$  to be semi-stable at  $s$ , there is an exact sequence

$$0 \rightarrow T \rightarrow A_s^\circ \rightarrow B \rightarrow 0$$

where  $T$  is a torus and  $B$  is an abelian variety and  $\dim(T) + \dim(B) = \dim(A_s) = \dim(A)$ . Now  $\dim_{\mathbb{F}_\ell}(T[\ell]) = \dim(T)$  and  $\dim_{\mathbb{F}_\ell}(B[\ell]) = 2 \dim(B) = 2 \dim(A) - 2 \dim(T)$ . Taking into account that we have an exact sequence

$$0 \rightarrow T[\ell] \rightarrow A_s^\circ[\ell] \rightarrow B[\ell] \rightarrow 0$$

(note that  $T(\tilde{k}) \cong (\tilde{k}^\times)^{\dim(T)}$  is divisible by  $\ell$ ), we find the relation  $\dim_{\mathbb{F}_\ell}(A_s[\ell]) = \dim_{\mathbb{F}_\ell}(A_s^\circ[\ell]) = 2 \dim(A) - \dim(T)$ . This implies c), because  $A_s[\ell] = A[\ell]^{I(\bar{v})}$  ([31, p. 495]) and obviously  $A[\ell]^{I(\bar{v})} = \text{Eig}(g, 1)$ .  $\square$

In general, if  $V$  is a finite dimensional vector space over  $\mathbb{F}_\ell$ , and  $g \in \text{End}_{\mathbb{F}_\ell}(V)$ , then one defines  $\text{drop}(g) = \dim(V) - \dim(\text{Eig}(g, 1))$ . One calls  $g$  a *transvection*, if it is unipotent of drop 1. We shall say that an abelian variety  $A$  over a field  $K$  is of *Hall type*, provided  $\text{End}(A) = \mathbb{Z}$  and there is a discrete valuation  $v$  on  $K$  such that  $A$  has semistable reduction of toric dimension 1 at  $v$  (i.e. at the maximal ideal of the discrete valuation ring of  $v$ ). We have thus proved the following

**Proposition 1.2.** *If  $A$  is an abelian variety of Hall type over a finitely generated field  $K$ , then there is a constant  $\ell_0$  such that  $\mathcal{M}_K(A[\ell])$  contains a transvection for every prime number  $\ell \geq \ell_0$ .*

## 2 Finiteness properties of division fields

If  $A$  is an abelian variety over a field  $K$  (of arbitrary characteristic) and  $p = \text{char}(K)$ , then we denote by  $A_{\neq p}$  the group of points in  $A(K_{\text{sep}})$  of order prime to  $p$ . Then

$$K(A_{\neq p}) = \prod_{\ell \neq p \text{ prime}} K(A[\ell^\infty]) = \bigcup_{n \notin p\mathbb{Z}} K(A[n]).$$

If  $p = 0$ , then  $K(A_{\neq p}) = K(A_{\text{tor}})$ . In this section we prove among other things: If  $K$  is finitely generated of positive characteristic, then  $G(K(A_{\neq p})/K)$  is a finitely generated profinite group.

In this section, a *function field of  $n$  variables* over a field  $F$  will be a finitely generated field extension  $E/F$  of transcendence degree  $n$ . As usual we call such a function field  $E/F$  of  $n$  variables *separable* if it has a separating transcendence base. The following Lemma is an easy consequence of [7, Proposition 3.1].

**Lemma 2.1.** *(cf. [7]) Let  $F$  be a field and  $K/F$  a function field of one variable. Assume that  $K/F$  is separable. Let  $p = \text{char}(F)$ . Let  $A/K$  be an abelian variety. Let  $F'$  be the algebraic closure of  $F$  in  $K(A_{\neq p})$ . Then  $G(K(A_{\neq p})/F'K)$  is a finitely generated profinite group.*

**Lemma 2.2.** *Let  $(K, v)$  be a discrete valued field,  $A/K$  an abelian variety with good reduction at  $v$ ,  $n$  an integer coprime to the residue characteristic of  $v$ ,  $L = K(A[n])$  and  $w$  an extension of  $v$  to  $L$ . Denote the residue field of  $v$  (resp.  $w$ ) by  $k(v)$  (resp.  $k(w)$ ). Let  $A_v/k(v)$  be the reduction of  $A$  at  $v$ . Then  $k(w) = k(v)(A_v[n])$ .*

*Proof.* Let  $R$  be the valuation ring of  $v$  and  $S = \text{Spec}(R)$ . Let  $\mathcal{A} \rightarrow S$  be an abelian scheme with generic fibre  $A$ . Then  $A_v = \mathcal{A} \times_S \text{Spec}(k(v))$ ,  $\mathcal{A}[n]$  is a finite étale group scheme over  $S$ , and if  $T$  is the normalization of  $S$  in  $L$ , then the restriction map  $r : A[n](L) \cong \mathcal{A}[n](T) \rightarrow A_v[n](k(w))$  is bijective (cf. [31]). The assertion follows easily from that.  $\square$

**Definition 2.3.** *We shall say in the sequel that a field  $K$  has property  $\mathcal{F}$ , if  $G(K'(A_{\neq p})/K')$  is a finitely generated profinite group for every finite separable extension  $K'/K$  and every abelian variety  $A/K'$ .*

**Proposition 2.4.** *Let  $F$  be a field that has property  $\mathcal{F}$ . Let  $p = \text{char}(F)$ . Let  $K$  be a function field over  $F$ . Assume that  $K/F$  is separable. Then  $K$  has property  $\mathcal{F}$ .*

*Proof.* By a routine induction on  $\text{trdeg}(K/F)$  it is enough to prove the Proposition in the special case where  $K/F$  is a function field in *one* variable. We may thus assume  $\text{trdeg}(K/F) = 1$  and we have to show that  $G(K'(A_{\neq p})/K')$  is finitely generated for every finite separable extension  $K'/K$  and every abelian variety  $A/K'$ . But if  $K'/K$  is a finite separable extension, then  $K'/F$  is a separable function field of one variable again. Hence it is enough to prove that  $G(K(A_{\neq p})/K)$  is finitely generated for every abelian variety  $A/K$ .

Let  $A/K$  be an abelian variety. Let  $F_0$  be the algebraic closure of  $F$  in  $K$ . Then  $K/F_0$  is a regular extension. Let  $C/F_0$  be a smooth curve with function field  $K$  and such that  $A$  has good reduction at all points of  $C$ . There is a finite Galois extension  $F_1/F_0$  such that  $C(F_1) \neq \emptyset$ . If we put  $K_1 := F_1K$ , then  $K_1/F_1$  is regular. Furthermore there is an exact sequence

$$1 \rightarrow G(K_1(A_{\neq p})/K_1) \rightarrow G(K(A_{\neq p})/K) \rightarrow G(K_1/K)$$

and  $G(K_1/K)$  is finite. If we prove that  $G(K_1(A_{\neq p})/K_1)$  is finitely generated, then it follows that  $G(K(A_{\neq p})/K)$  is finitely generated as well. Hence we may assume that  $K_1 = K$ , i.e. that  $K/F$  is regular and that  $C(F) \neq \emptyset$ .

Choose a point  $c \in C(F)$  and denote by  $A_c/F$  the (good) reduction of  $A$  at  $c$ . As in Lemma 2.1 denote by  $F'$  the algebraic closure of  $F$  in  $K(A_{\neq p})$ .

**Claim.**  $F' \subset F(A_{c, \neq p})$ .

Let  $x \in F'$ . Then  $x$  is algebraic over  $F$  and  $x \in K(A[n])$  for some  $n$  which is coprime to  $p$ . If  $F_n$  denotes the algebraic closure of  $F$  in  $K(A[n])$ , then  $x \in F_n$ . Let  $w$  be the extension to  $K(A[n])$  of the valuation attached to  $c$ . Then  $k(w) = F(A_c[n])$  by Lemma 2.2. Obviously  $F_n \subset k(w)$ . Hence  $x \in F(A_c[n]) \subset F(A_{c, \neq p})$ . This finishes the proof of the Claim.

The profinite group  $G(F(A_{c,\neq p})/F)$  is finitely generated, because  $F$  has property  $\mathcal{F}$  by assumption. Hence its quotient  $G(F'/F)$  is finitely generated as well. Note that  $G(F'K/K) = G(F'/F)$ . On the other hand  $G(K(A_{\neq p})/F'K)$  is finitely generated by Lemma 2.1. From the exact sequence

$$1 \rightarrow G(K(A_{\neq p})/F'K) \rightarrow G(K(A_{\neq p})/K) \rightarrow G(F'K/K) \rightarrow 1$$

we see that  $G(K(A_{\neq p})/K)$  is finitely generated as desired.  $\square$

**Corollary 2.5.** *Let  $K$  be a finitely generated field of positive characteristic or  $K$  be a function field over an algebraically closed field of arbitrary characteristic. Then  $K$  has property  $\mathcal{F}$ . In particular  $G(K(A_{\neq p})/K)$  is finitely generated for every abelian variety  $A/K$ .*

*Proof.* In both cases  $K$  is a function field over a perfect field  $F$  which has property  $\mathcal{F}$ . The assertion hence follows from Proposition 2.4  $\square$

**Remark 2.6.** *A finitely generated field  $K$  of characteristic zero does not have property  $\mathcal{F}$ . In fact, if  $A/K$  is principally polarized abelian variety, then by the existence of the Weil pairing  $K(A_{\text{tor}}) \supset K(\mu_\infty)$ , and plainly  $G(K(\mu_\infty)/K)$  is not finitely generated, when  $K$  is a finitely generated extension of  $\mathbb{Q}$ .*

### 3 Monodromy Computations

Let  $K$  be a field and  $A/K$  an abelian variety. We begin with the question whether  $A[\ell]$  is a simple  $G_K$ -module for sufficiently large  $\ell$ .

**Proposition 3.1.** *Let  $A$  be an abelian variety over a finitely generated field  $K$ . Assume that  $\text{End}_K(A) = \mathbb{Z}$ . Then there is a constant  $\ell_0$  such that  $A[\ell]$  is a simple  $\mathbb{F}_\ell[G_K]$ -module for all primes  $\ell \geq \ell_0$ .*

In the cases we need to consider, this Proposition is a consequence of the following classical result (cf. [6, p. 118, p. 204], [34], [35],[22]).

**Theorem 3.2. (Faltings, Zarhin)** *Let  $K$  be a finitely generated field and  $A/K$  an abelian variety. Then there is a constant  $\ell_0 > \text{char}(K)$  such that the  $\mathbb{F}_\ell[G_K]$ -module  $A[\ell]$  is semisimple and the canonical map  $\text{End}_K(A) \otimes \mathbb{F}_\ell \rightarrow \text{End}_{\mathbb{F}_\ell}(A[\ell])$  is injective with image  $\text{End}_{\mathbb{F}_\ell[G_K]}(A[\ell])$  for all primes  $\ell \geq \ell_0$ .*

*Proof of Proposition 3.1.* By Theorem 3.2 there is a constant  $\ell_0$  such that  $A[\ell]$  is a semisimple  $\mathbb{F}_\ell[G_K]$ -module with  $\text{End}_{\mathbb{F}_\ell[G_K]}(A[\ell]) = \mathbb{F}_\ell \text{Id}$  for every prime  $\ell \geq \ell_0$ . This is only possible if  $A[\ell]$  is a simple  $\mathbb{F}_\ell[G_K]$ -module for all primes  $\ell \geq \ell_0$ .  $\square$

We need some notation in order to explain a theorem of Raynaud that will be of importance later. Let  $E/\mathbb{F}_p$  be a finite field extension with  $|E| = p^d$  and  $F/\mathbb{F}_p$  an algebraic extension. Denote by  $\text{Emb}(E, \tilde{F})$  the set of all embeddings

$E \rightarrow \tilde{F}$ . If  $i \in \text{Emb}(E, \tilde{F})$  is one such embedding, then there is a unique function  $e : \text{Emb}(E, \tilde{F}) \rightarrow \{0, \dots, p-1\}$  such that

$$\chi = \prod_{j \in \text{Emb}(E, \tilde{F})} (j|E^\times)^{e(j)},$$

and such that  $e(j) < p-1$  for some  $j \in \text{Emb}(E, \tilde{F})$ . We define  $\text{amp}(\chi) := \max(e(j) : j \in \text{Emb}(E, \tilde{F}))$  to be the *amplitude of the character*  $\chi$ . Let  $\rho : E^\times \rightarrow \text{Aut}_{\mathbb{F}_p}(V)$  be a representation of  $E^\times$  on a finite dimensional  $\mathbb{F}_p$ -vector space  $V$ . If  $V$  is a *simple*  $\mathbb{F}_p[E^\times]$ -module, then there is a finite field  $F_V$  with  $|F_V| = |V|$  and a structure of 1-dimensional  $F_V$ -vector space on  $V$  such that  $\rho$  factors through a character  $\chi_\rho : E^\times \rightarrow F_V^\times$ . We then define  $\text{amp}(V) := \text{amp}(\rho) := \text{amp}(\chi_\rho)$ . In general we define  $\text{amp}(V) := \text{amp}(\rho) := \max(\text{amp}(V_i) : i = 1, \dots, t)$  where  $\{V_1, \dots, V_t\}$  is the set of Jordan-Hölder quotients of  $V$  to be the *amplitude of the representation*  $\rho$ .

**Theorem 3.3. (Raynaud [24], [26, p. 277])** *Let  $A$  be an abelian variety over a number field  $K$ . Let  $v$  be a place of  $K$  with residue characteristic  $p$ . Let  $e$  be the ramification index of  $v|\mathbb{Q}$ . Let  $w$  be an extension of  $v$  to  $K(A[p])$ . Let  $I$  be the inertia group of  $w|v$  and  $P$  the  $p$ -Sylow subgroup of  $I$ . Let  $C \subset I$  be a subgroup that maps isomorphically onto  $I/P$ . Then there is a finite extension  $E/\mathbb{F}_p$  and a surjective homomorphism  $E^\times \rightarrow C$  such that the resulting representation*

$$\rho : E^\times \rightarrow C \rightarrow \text{Aut}_{\mathbb{F}_p}(A[p])$$

*has amplitude  $\text{amp}(\rho) \leq e$ .*

The technical heart of our monodromy computations is the following group theoretical result, which can be extracted from the work of C. Hall [14], [15].

**Theorem 3.4.** *Let  $\ell > 2$  be a prime, let  $(V, e_V)$  be a finite-dimensional symplectic space over  $\mathbb{F}_\ell$  and  $M$  a subgroup of  $\Gamma := \text{GSp}(V, e_V)$ . Assume that  $M$  contains a transvection and that  $V$  is a simple  $\mathbb{F}_\ell[M]$ -module. Denote by  $R$  the subgroup of  $M$  generated by the transvections in  $M$ .*

*a) Then there is a non-zero symplectic subspace  $W \subset V$ , which is a simple  $\mathbb{F}_\ell[R]$ -module, such that the following properties hold true:*

- i) Let  $H = \text{Stab}_M(W)$ . There is a orthogonal direct sum decomposition  $V = \bigoplus_{g \in M/H} gW$ . In particular  $|M/H| \leq \dim(V)$ .*
- ii)  $R \cong \prod_{g \in M/H} \text{Sp}(W)$  and  $N_\Gamma(R) \cong \prod_{g \in M/H} \text{GSp}(W) \rtimes \text{Sym}(M/H)$ .*
- iii)  $R \subset M \subset N_\Gamma(R)$ .*

*Denote by  $\varphi : N_\Gamma(R) \rightarrow \text{Sym}(M/H)$  the projection.*

*b) Let  $e \in \mathbb{N}$ . Let  $E/\mathbb{F}_\ell$  be a finite extension and  $\rho : E^\times \rightarrow M \subset \text{GSp}(V, e_V)$  a homomorphism such that the corresponding representation of  $E^\times$  on  $V$  has amplitude  $\text{amp}(\rho) \leq e$ . If  $\ell > \dim(V)e + 1$ , then  $\varphi(\rho(E^\times)) = \{1\}$ .*

Hall's proof in [14], [15] addresses a slightly less general situation. We will present a self-contained proof of Theorem 3.4 in the Appendix.

**Remark 3.5.** *Assume that in the situation of Theorem 3.4 the module  $V$  is a simple  $\mathbb{F}_\ell[\ker(\varphi) \cap M]$ -module. Then  $V$  is in particular a simple  $\mathbb{F}_\ell[\ker(\varphi)]$ -module and  $\ker(\varphi) = \prod_{g \in M/H} \mathrm{GSp}(W)$ . This is only possible if  $M = H$ ,  $V = W$  and  $R = \mathrm{Sp}(V, e) \subset M$ .*

We now state the main result of this section.

**Theorem 3.6.** *Let  $K$  be a finitely generated field. Let  $(A, \lambda)$  be a polarized abelian variety over  $K$  of Hall type. Then  $(A, \lambda)$  has big monodromy.*

The case where  $K$  is a global field is due to Hall (cf. [15]) and we follow his line of proof to some extent, but we need a lot of additional arguments in order to make things work in the more general situation. The proof will occupy almost the rest of this section.

There is a constant  $\ell_0 > \max(\deg(\lambda), \mathrm{char}(K))$  such that the following holds true for all primes  $\ell \geq \ell_0$ :

1. The subgroup  $\mathcal{M}_K(A[\ell])$  of  $\mathrm{GSp}(A[\ell], e_\ell^\lambda)$  contains a transvection. Denote by  $R_\ell$  the subgroup of  $\mathcal{M}_K(A[\ell])$  generated by the transvections in  $\mathcal{M}_K(A[\ell])$  (cf. Proposition 1.2).
2.  $A[\ell]$  is a simple  $\mathbb{F}_\ell[G_K]$ -module (cf. Proposition 3.1).

Now Hall's group theory result (cf. Theorem 3.4) gives - for every prime  $\ell \geq \ell_0$  - a non-zero symplectic subspace  $W_\ell \subset A[\ell]$ , which is simple as a  $\mathbb{F}_\ell[R_\ell]$ -module such that the properties i), ii) and iii) of Theorem 3.4 are satisfied. Let  $H_\ell$  be the stabilizer of  $W_\ell$  under the action of  $\mathcal{M}_K(A[\ell])$ . Define  $M_\ell := \mathcal{M}_K(A[\ell])$  and  $\Gamma_\ell := \mathrm{GSp}(A[\ell], e_\ell^\lambda)$ . Then

$$\prod_{M_\ell/H_\ell} \mathrm{Sp}(W_\ell, e_\ell^\lambda) \cong R_\ell \subset M_\ell \subset N_{\Gamma_\ell}(R_\ell) = \prod_{M_\ell/H_\ell} \mathrm{Sp}(W_\ell, e_\ell^\lambda) \rtimes \mathrm{Sym}(M_\ell/H_\ell),$$

and we denote by  $\varphi_\ell : N_{\Gamma_\ell}(R_\ell) \rightarrow \mathrm{Sym}(M_\ell/H_\ell)$  the projection. We have the following property (cf. Remark 3.5):

*If  $A[\ell]$  is a simple  $\mathbb{F}_\ell[\ker(\varphi_\ell) \cap M_\ell]$ -module for some prime  $\ell \geq \ell_0$ , then  $M_\ell = H_\ell$ ,  $W_\ell = A[\ell]$  and  $M_\ell \supset \mathrm{Sp}(A[\ell], e_\ell^\lambda)$  for this prime  $\ell$ .*

We denote by  $N_\ell$  the fixed field inside  $K_{\mathrm{sep}}$  of the preimage  $\rho_{A[\ell]}^{-1}(M_\ell \cap \ker(\varphi_\ell))$ , where  $\rho_{A[\ell]} : G_K \rightarrow \Gamma_\ell$  is the mod- $\ell$  representation attached to  $A$ . Then  $N_\ell$  is an intermediate field of  $K(A[\ell])/K$  which is Galois over  $K$ , and  $G(N_\ell/K)$  is isomorphic to the subgroup  $\varphi_\ell(M_\ell)$  of  $\mathrm{Sym}(M_\ell/H_\ell)$ . In particular  $[N_\ell : K] \leq (2 \dim(A))!$  is bounded independently of  $\ell$ . If we denote by  $N := \prod_{\ell \geq \ell_0, \text{ prime}} N_\ell$  the corresponding composite field, then  $G_N = \bigcap_{\ell \geq \ell_0, \text{ prime}} G_{N_\ell}$ . Hence the following property holds true.

*If  $A[\ell]$  is simple as a  $\mathbb{F}_\ell[G_N]$ -module for some prime  $\ell \geq \ell_0$ , then  $M_\ell \supset \mathrm{Sp}(A[\ell], e_\ell^\lambda)$  for this prime  $\ell$ . (\*)*

*Proof of Theorem 3.6 in the special case  $\text{char}(K) > 0$ .* If  $\text{char}(K) > 0$ , then the Galois group  $G(K(A_{\neq p})/K)$  ( $p := \text{char}(K)$ ) is finitely generated, because  $K$  then has property  $\mathcal{F}$  by Corollary 2.5. Furthermore  $N_\ell$  is an intermediate field of  $K(A_{\neq p})/K$  which is Galois over  $K$  and with  $[N_\ell : K]$  bounded independently of  $\ell$ . Hence  $N/K$  must be finite. In particular  $N$  is finitely generated. A second application of the result of Faltings and Zarhin (cf. Proposition 3.1) yields a constant  $\ell_1 \geq \ell_0$  such that  $A[\ell]$  is a simple  $\mathbb{F}_\ell[G_N]$ -module for all primes  $\ell \geq \ell_0$ . Hence  $A$  has big monodromy by (\*).  $\square$

To finish the proof of Theorem 3.6 we assume for the rest of the proof that  $\text{char}(K) = 0$ . We shall prove that  $N/K$  is finite also in that case, but the proof of this fact is more complicated, because now  $K$  is *not*  $\mathcal{F}$ -finite (cf. Remark 2.6). We briefly sketch the main steps in the proof, before we go into the details: The first and hardest step is to show that the algebraic closure  $L$  of  $\mathbb{Q}$  in  $N$  is a *finite* extension of  $\mathbb{Q}$ . In order to achieve this we will construct a finite extension  $L'/\mathbb{Q}$  such that some  $L'$ -rational “place” of  $KL'$  splits up completely into  $L'$ -rational “places” of  $N_\ell L'$  for every sufficiently large prime  $\ell$ . We use this to show that  $G(NL/KL) \cong G(NL_{\text{sep}}/KL_{\text{sep}})$  and the fact that the latter group can be proved to be finite, because  $KL_{\text{sep}}$  is  $\mathcal{F}$ -finite (unlike  $K$  itself). This suffices to prove that  $N/K$  is finite. Once we know this, we shall proceed as in the positive characteristic case above.

We now go into the details. Let  $F$  be the algebraic closure of  $\mathbb{Q}$  in  $K$ . Then  $F$  is a number field. Let  $S$  be a smooth affine  $F$ -variety with function field  $K$  such that  $A$  extends to an abelian scheme  $\mathcal{A}$  over  $S$  with generic fibre  $A$  (i.e. such that  $A$  has good reduction along  $S$ ). Let  $S_\ell$  be the normalization of  $S$  in  $N_\ell$  and let  $S'_\ell$  be the normalization of  $S_\ell$  in  $K(A[\ell])$ . Then  $S'_\ell \rightarrow S_\ell \rightarrow S$  are finite étale covers. (Note that  $\text{char}(F(s)) = 0$  for every point  $s \in S$ .) In particular  $S'_\ell$  and  $S_\ell$  are smooth  $F$ -schemes. (Compare the diagram below.)

Fix a geometric point  $P \in S(F_{\text{sep}})$  and denote by  $A_P := \mathcal{A} \times_S \text{Spec}(F(P))$  the corresponding special fibre of  $\mathcal{A}$ . Then  $A_P$  is an abelian variety over the number field  $F(P)$ . Fix for every  $\ell \geq \ell_0$  a geometric point  $Q_\ell \in S_\ell(F_{\text{sep}})$  over  $P$  and a geometric point  $Q'_\ell \in S'_\ell(F_{\text{sep}})$  over  $Q_\ell$ . Then  $F(Q'_\ell)/F(Q_\ell)$  and  $F(Q_\ell)/F(P)$  are finite extensions of number fields. Note that  $F(Q'_\ell) = F(P)(A_P[\ell])$  by Lemma 2.2. Denote by  $\mathcal{O}$  (resp.  $\mathcal{O}_\ell$ , resp.  $\mathcal{O}'_\ell$ ) the integral closure of  $\mathbb{Z}$  in  $F(P)$  (resp. in  $F(Q_\ell)$ , resp. in  $F(Q'_\ell)$ ). For every prime  $\ell \geq \ell_0$  we have the following diagram on the level of schemes

$$\begin{array}{ccccc}
\mathrm{Spec}(K(A[\ell])) & \longrightarrow & \mathrm{Spec}(N_\ell) & \longrightarrow & \mathrm{Spec}(K) \\
\downarrow & & \downarrow & & \downarrow \\
S'_\ell & \longrightarrow & S_\ell & \longrightarrow & S \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Spec}(F(P)(A_P[\ell])) & \xlongequal{\quad} & \mathrm{Spec}(F(Q'_\ell)) & \longrightarrow & \mathrm{Spec}(F(Q_\ell)) & \longrightarrow & \mathrm{Spec}(F(P)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(\mathcal{O}'_\ell) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_\ell) & \xrightarrow{f_\ell} & \mathrm{Spec}(\mathcal{O})
\end{array}$$

We now study the ramification of prime ideals  $\mathfrak{m} \in \mathrm{Spec}(\mathcal{O})$  in the extension  $F(Q_\ell)/F(P)$ . Let  $\mathbb{P}_{\mathrm{bad}}$  be the (finite) set of primes  $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$  where  $A_P/F(P)$  has bad reduction.

**Lemma 3.7.** *There is a constant  $\ell_2 \geq \ell_0$  with the following property: For every prime number  $\ell \geq \ell_2$  the map  $f_\ell : \mathrm{Spec}(\mathcal{O}_\ell) \rightarrow \mathrm{Spec}(\mathcal{O})$  is étale at every point  $\mathfrak{m} \in \mathrm{Spec}(\mathcal{O})$  outside of  $\mathbb{P}_{\mathrm{bad}}$ .*

*Proof.* Let  $\ell_2 := \max(\ell_0, (2 \dim(A))! [F(P) : \mathbb{Q}] + 2)$ .

Now let  $\ell \geq \ell_2$  be a prime number. Let  $\mathfrak{m} \in \mathrm{Spec}(\mathcal{O})$  be an arbitrary prime ideal with  $\mathfrak{m} \notin \mathbb{P}_{\mathrm{bad}}$ . We have to show that  $\mathfrak{m}$  is unramified in  $F(Q_\ell)$ . Let  $p = \mathrm{char}(\mathcal{O}/\mathfrak{m})$  be the residue characteristic of  $\mathfrak{m}$ .

If  $p \neq \ell$ , then  $\mathfrak{m}$  is unramified even in  $F(Q'_\ell) = F(P)(A_P[\ell])$ .

We can hence assume that  $\boxed{p = \ell}$ . Let  $\mathfrak{m}_\ell \in \mathrm{Spec}(\mathcal{O}_\ell)$  be a point over  $\mathfrak{m}$  and  $\mathfrak{m}'_\ell \in \mathrm{Spec}(\mathcal{O}'_\ell)$  a point over  $\mathfrak{m}_\ell$ . Let  $D(\mathfrak{m}'_\ell)$  (resp.  $D(\mathfrak{m}_\ell)$ ) be the decomposition group of  $\mathfrak{m}'_\ell/F(P)$  (resp. of  $\mathfrak{m}_\ell/F(P)$ ) and  $I(\mathfrak{m}'_\ell)$  (resp.  $I(\mathfrak{m}_\ell)$ ) the corresponding inertia group. Let  $P(\mathfrak{m}'_\ell)$  (resp.  $P(\mathfrak{m}_\ell)$ ) be the (unique)  $p$ -Sylow subgroup of  $I(\mathfrak{m}'_\ell)$  (resp.  $I(\mathfrak{m}_\ell)$ ).

We have the following commutative diagram on the level of groups:

$$\begin{array}{ccccc}
\prod_{M_\ell/H_\ell} \mathrm{GSp}(W_\ell) \hookrightarrow & N_{\Gamma_\ell}(M_\ell) & \twoheadrightarrow & \mathrm{Sym}(M_\ell/H_\ell) \\
\uparrow & \uparrow & & \uparrow \\
M_\ell \cap \ker(\varphi_\ell) \hookrightarrow & M_\ell & \twoheadrightarrow & \varphi_\ell(M_\ell) \\
\parallel & \parallel & & \parallel \\
G(K(A[\ell])/N_\ell) \hookrightarrow & G(K(A[\ell])/K) & \twoheadrightarrow & G(N_\ell/K) \\
\uparrow & \uparrow & & \uparrow \\
G(F(Q'_\ell)/F(Q_\ell)) \hookrightarrow & G(F(Q'_\ell)/F(P)) & \twoheadrightarrow & G(F(Q_\ell)/F(P)) \\
& \uparrow & & \uparrow \\
& D(\mathfrak{m}'_\ell) & \twoheadrightarrow & D(\mathfrak{m}_\ell) \\
& \uparrow & & \uparrow \\
& I(\mathfrak{m}'_\ell) & \twoheadrightarrow & I(\mathfrak{m}_\ell) \\
& \uparrow & & \uparrow \\
& P(\mathfrak{m}'_\ell) & \twoheadrightarrow & P(\mathfrak{m}_\ell)
\end{array}$$

We have to prove that the image of  $I(\mathfrak{m}'_\ell)$  in  $\mathrm{Sym}(M_\ell/H_\ell)$  by the maps in the diagram is  $\{1\}$ . Now  $p = \ell > (2 \dim(A))!$  due to our choice of  $\ell_2$  and  $|\mathrm{Sym}(M_\ell/H_\ell)| \leq (2 \dim(A))!$ , hence  $P(\mathfrak{m}'_\ell)$  maps to  $\{1\}$  in  $\mathrm{Sym}(M_\ell/H_\ell)$ . In particular,  $P(\mathfrak{m}_\ell) = \{1\}$ . Consider the tame ramification group  $I_t = I(\mathfrak{m}'_\ell)/P(\mathfrak{m}'_\ell)$ . It is a cyclic group of order prime to  $p$ . Choose a subgroup  $C \subset I(\mathfrak{m}'_\ell)$  that maps isomorphically onto  $I_t$  under the projection. It is enough to show that  $C$  maps to  $\{1\}$  in  $\mathrm{Sym}(M_\ell/H_\ell)$ .

By Raynaud's theorem (cf. Theorem 3.3) there is a finite extension  $E/\mathbb{F}_p$  and an epimorphism  $E^\times \rightarrow C$  such that the resulting representation

$$E^\times \rightarrow C \rightarrow \mathrm{Aut}(A_P[\ell]) = \mathrm{Aut}(A[\ell])$$

has amplitude  $\leq e$ , where  $e$  is the ramification index of  $\mathfrak{m}$  over  $\mathbb{Q}$ . Clearly  $e \leq [F(P) : \mathbb{Q}]$ . By part b) of Theorem 3.4, the image of  $E^\times$  in  $\mathrm{Sym}(M_\ell/H_\ell)$  is  $\{1\}$ . Hence the image of  $C$  in  $\mathrm{Sym}(M_\ell/H_\ell)$  is  $\{1\}$  as desired.  $\square$

**Lemma 3.8.** *Let  $L$  be the algebraic closure of  $F$  in  $N$ . Then  $L/F$  is a finite extension.*

*Proof.* Let  $L' := \prod_{\ell > \ell_0, \text{ prime}} F(Q_\ell)$ . For every prime  $\ell \geq \ell_2$  the Galois extension of number fields  $F(Q_\ell)/F(P)$  is unramified outside  $\mathbb{P}_{\mathrm{bad}}$  by Lemma 3.7. Furthermore  $[F(Q_\ell) : F(P)] \leq (2 \dim(A))!$  for every prime  $\ell \geq \ell_2$ . The Theorem of Hermite-Minkowski (cf. [19], p. 122) implies that  $\prod_{\ell \geq \ell_2, \text{ prime}} F(Q_\ell)$  is a finite extension of  $F(P)$ . This in turn implies that  $L'/F$  is a finite extension. It is thus enough to show that  $L \subset L'$ .

Recall that  $K = F(S)$  is the function field of the  $F$ -variety  $S$  and  $S_\ell$  is the normalization of  $S$  in the finite Galois extension  $N_\ell/K$ . Denote by  $\hat{S}$  the normalization of  $S$  in  $N$  and by  $h_\ell : \hat{S} \rightarrow S_\ell$  the canonical projection. The canonical morphism  $\hat{S} \rightarrow S$  is surjective, hence there is a point  $\hat{P} \in \hat{S}(F_{\text{sep}})$  over  $P$ . The point  $h_\ell(\hat{P}) \in S_\ell(F_{\text{sep}})$  lies over  $P$ . Hence  $h_\ell(\hat{P})$  is conjugate to  $Q_\ell$  under the action of  $G(N_\ell/K)$ . This implies that  $F(h_\ell(\hat{P})) = F(Q_\ell)$ . For every  $\ell \geq \ell_0$  there is a diagram

$$\begin{array}{ccccc}
\text{Spec}(N) & \longrightarrow & \text{Spec}(N_\ell) & \longrightarrow & \text{Spec}(K) \\
\uparrow & & \uparrow & & \uparrow \\
\hat{S} & \xrightarrow{h_\ell} & S_\ell & \longrightarrow & S \\
\downarrow & & \downarrow & & \downarrow \\
\text{Spec}(F(\hat{P})) & \longrightarrow & \text{Spec}(F(Q_\ell)) & \longrightarrow & \text{Spec}(F(P))
\end{array}$$

where the morphisms  $S_\ell \rightarrow S$  are étale covers and  $N = \prod_{\ell \geq \ell_0} N_\ell$ . It follows that  $F(\hat{P}) = \prod_{\ell \geq \ell_0} F(Q_\ell) = L'$ . On the other hand  $L$  is the algebraic closure of  $F$  in  $N$ , hence  $\hat{S}$  is a scheme over  $L$ . This implies that  $L$  is a subfield of  $F(\hat{P})$ . Hence in fact  $L \subset L'$  as desired.  $\square$

*End of the proof of Theorem 3.6 in the case  $\text{char}(K) = 0$ .* We have an isomorphism  $G(NL_{\text{sep}}/KL_{\text{sep}}) \cong G(N/KL)$ , because  $N/L$  and  $KL/L$  are regular extensions. The field  $KL_{\text{sep}}$  is  $\mathcal{F}$ -finite by Corollary 2.5. Hence the profinite group  $G(KL_{\text{sep}}(A_{\text{tor}})/KL_{\text{sep}})$  is finitely generated. As  $NL_{\text{sep}} \subset KL_{\text{sep}}(A_{\text{tor}})$ ,  $G(NL_{\text{sep}}/KL_{\text{sep}})$  must be finitely generated as well. Furthermore  $NL_{\text{sep}} = \prod_{\ell \geq \ell_0} N_\ell L_{\text{sep}}$  where  $[N_\ell L_{\text{sep}} : KL_{\text{sep}}]$  is bounded independently from  $\ell$ . Hence  $G(NL_{\text{sep}}/KL_{\text{sep}})$  is finite and this implies that  $N/KL$  is a finite extension. On the other hand it follows from Lemma 3.8 that  $KL/K$  is finite. Hence  $N/K$  is a *finite* extension. Consequently  $N$  is finitely generated, because  $K$  is finitely generated. Proposition 3.1 yields a constant  $\ell_3 > \ell_0$  such that  $A[\ell]$  is a simple  $\mathbb{F}_\ell(G_N)$ -module for every prime  $\ell \geq \ell_3$ . Hence  $A/K$  has big monodromy by  $(*)$ , as desired.  $\square$

## 4 Applications

In this section we apply our methods to prove a generalization of a result of Ellenberg, Elsholz, Hall and Kowalski on endomorphism rings and simplicity of fibres in certain families of abelian varieties (cf. [5, Theorem 8]).

**Proposition 4.1.** *Let  $K$  be a field and  $(A, \lambda)$  a polarized abelian variety over  $K$  with big monodromy. Let  $L/K$  be a finite extension. Then the following properties hold.*

- a) *There is a constant  $\ell_0 \geq \max(\text{char}(K), \deg(\lambda))$  such that  $\mathcal{M}_L(A[\ell]) \supset \text{Sp}(A[\ell], e_\ell^\lambda)$  for every prime number  $\ell \geq \ell_0$ .*

b)  $A$  is geometrically simple.

*Proof.* Part a). Let  $E_0$  be the maximal separable extension of  $K$  in  $L$  and  $E/K$  a finite Galois extension containing  $E_0$ . By our assumption there is a constant  $\ell_0 > \max(\deg(\lambda), \text{char}(K), 5)$  such that  $\mathcal{M}_K(A[\ell]) \supset \text{Sp}(A[\ell], e_\ell^\lambda)$  for every prime  $\ell \geq \ell_0$ . For  $\ell \geq \ell_0$  let  $K_\ell$  be the fixed field of  $\text{Sp}(A[\ell], e_\ell^\lambda)$  in  $K(A[\ell])/K$ . Then  $\mathcal{M}_{K_\ell}(A[\ell]) = \text{Sp}(A[\ell], e_\ell^\lambda)$  and  $\mathcal{M}_{EK_\ell}(A[\ell])$  is a normal subgroup of  $\mathcal{M}_{K_\ell}(A[\ell])$  of index  $\leq [E : K]$ . Put  $\ell_1 := \max(\ell_0, [E : K] + 1)$ . Then

$$|\mathcal{M}_{EK_\ell}(A[\ell])| \geq \frac{1}{[E : K]} |\text{Sp}(A[\ell], e_\ell^\lambda)| > 2$$

for all primes  $\ell \geq \ell_1$ . On the other hand the only proper normal subgroups of  $\text{Sp}(A[\ell], e_\ell^\lambda)$  are  $\{\pm 1\}$  and the trivial group (cf. [30, p. 53]). Hence

$$\mathcal{M}_{E_0}(A[\ell]) \supset \mathcal{M}_E(A[\ell]) \supset \mathcal{M}_{EK_\ell}(A[\ell]) = \text{Sp}(A[\ell], e_\ell^\lambda)$$

for all primes  $\ell \geq \ell_1$ . As  $L/E_0$  is purely inseparable, we find

$$\mathcal{M}_L(A_L[\ell]) = \mathcal{M}_{E_0}(A[\ell]) \supset \text{Sp}(A[\ell], e_\ell^\lambda)$$

for all primes  $\ell \geq \ell_1$  as desired.

Part b). Let  $A_1, A_2/\tilde{K}$  be abelian varieties and  $f : A_{\tilde{K}} \rightarrow A_1 \times A_2$  an isogeny. Then  $A_1, A_2$  and  $f$  are defined over some finite extension  $L/K$ . Hence there is an  $\mathbb{F}_\ell[G_L]$ -module isomorphism  $A[\ell] \cong A_1[\ell] \times A_2[\ell]$  for every prime  $\ell > \deg(f)$ . By part a)  $\mathcal{M}_L(A[\ell]) \supset \text{Sp}(A[\ell], e_\ell^\lambda)$  for all sufficiently large primes  $\ell$ . Hence  $A[\ell]$  is a simple  $\mathbb{F}_\ell[\mathcal{M}_L(A[\ell])]$ -module and in particular a simple  $\mathbb{F}_\ell(G_L)$ -module for all sufficiently large primes  $\ell$ . This is only possible if  $A_1 = 0$  or  $A_2 = 0$ .  $\square$

Let  $F$  be a finitely generated field and  $K/F$  a finitely generated transcendental field extension and  $A/K$  an abelian variety. We say that  $A/K$  is *weakly isotrivial with respect to  $F$* , if there is an abelian variety  $B/\tilde{F}$  and a  $\tilde{K}$ -isogeny  $B_{\tilde{K}} \rightarrow A_{\tilde{K}}$ .

**Proposition 4.2.** *Let  $F$  be a finitely generated field,  $K/F$  a finitely generated separable transcendental field extension and  $(A, \lambda)$  a polarized abelian variety over  $K$ . Assume that  $A/K$  has big monodromy and that  $A/K$  is not weakly isotrivial with respect to  $F$ . Define  $K' := F_{\text{sep}}K$ . Then there is a constant  $\ell_0 \geq \max(\text{char}(K), \deg(\lambda))$  such that  $\mathcal{M}_{K'}(A[\ell]) = \text{Sp}(A[\ell], e_\ell^\lambda)$  for every prime number  $\ell \geq \ell_0$ .*

*Proof.* Let  $\ell_0 \geq \max(\deg(\lambda), \text{char}(K), 5)$  be a constant such that  $\mathcal{M}_K(A[\ell]) \supset \text{Sp}(A[\ell], e_\ell^\lambda)$  for every prime  $\ell \geq \ell_0$ . Let  $\ell \geq \ell_0$  be a prime number. Then we have

$$\mathcal{M}_{K'}(A[\ell]) \subset \text{Sp}(A[\ell], e_\ell^\lambda) \subset \mathcal{M}_K(A[\ell]),$$

because  $K'$  contains  $\mu_\ell$ . Furthermore  $\mathcal{M}_{K'}(A[\ell])$  a normal subgroup of  $\mathcal{M}_K(A[\ell])$ , because  $K'/K$  is Galois. It follows that  $\mathcal{M}_{K'}(A[\ell])$  is *normal* in  $\text{Sp}(A[\ell], e_\ell^\lambda)$ .

The only proper normal subgroups in  $\text{Sp}(A[\ell], e_\ell^\lambda)$  are  $\{1\}$  and  $\{\pm 1\}$  (cf. [30, p. 53]), because  $\ell \geq 5$ . Hence either  $\mathcal{M}_{K'}(A[\ell]) = \text{Sp}(A[\ell], e_\ell^\lambda)$  or  $|\mathcal{M}_{K'}(A[\ell])| \leq 2$ .

Let  $\Lambda$  be the set of prime numbers  $\ell \geq \ell_0$  where  $|\mathcal{M}_{K'}(A[\ell])| \leq 2$ . We claim that  $\Lambda$  is *finite*.

For every  $\ell \in \Lambda$  we have  $[K'(A[\ell]) : K'] \leq 2$ . Furthermore  $G(K'(A_{\neq p})/K')$  is profinitely generated, where  $p = \text{char}(K)$ . To see this note that

$$G(K'(A_{\neq p})/K') = G(\tilde{F}K'(A_{\neq p})/\tilde{F}K')$$

because  $\tilde{F}/F_{\text{sep}}$  is purely inseparable and use Corollary 2.5. Hence  $N := \prod_{\ell \in \Lambda} K'(A[\ell])$  is a *finite* extension of  $K'$ . In particular  $N/F_{\text{sep}}$  is a finitely generated regular extension.  $A/K$  must be geometrically simple by our assumption that  $A/K$  has big monodromy (cf. Proposition 4.1). In particular  $A_N$  is simple. Hence assumption that  $A$  is not weakly isotrivial with respect to  $F$  implies that the Chow trace  $\text{Tr}_{N/F_{\text{sep}}}(A_N)$  is zero. It follows by the Mordell-Lang-Néron theorem (cf. [3, Theorem 2.1]) that  $A(N)$  is a finitely generated  $\mathbb{Z}$ -module. In particular the torsion group  $A(N)_{\text{tor}}$  is finite. On the other hand,  $A(N)$  contains a non-trivial  $\ell$ -torsion point for every  $\ell \in \Lambda$ . It follows that  $\Lambda$  is in fact finite.

Thus, after replacing  $\ell_0$  by a bigger constant, we see that  $\mathcal{M}_{K'}(A[\ell]) = \text{Sp}(A[\ell], e_\ell^\lambda)$  for all primes  $\ell \geq \ell_0$ .  $\square$

**Corollary 4.3.** *Let  $F$  be a finitely generated field and  $K = F(t)$  the function field of  $\mathbb{P}^1/F$ . Let  $A/K$  be a polarized abelian variety. Let  $U \subset \mathbb{P}^1$  be an open subscheme such that  $A$  extends to an abelian scheme  $\mathcal{A}/U$ . For  $u \in U(F)$  denote by  $A_u/F$  the corresponding special fiber of  $\mathcal{A}$ . Assume that  $A$  is not weakly isotrivial with respect to  $F$  and that either condition i) or ii) is satisfied.*

- i)  $A$  is of Hall type.
- ii)  $\text{char}(K) = 0$ ,  $\text{End}(A) = \mathbb{Z}$  and  $\dim(A) = 2, 6$  or odd.

Then the sets:

$$X_1 := \{u \in U(F) \mid \text{End}(A_u) \neq \mathbb{Z}\}$$

and

$$X_2 := \{u \in U(F) \mid A_u/F \text{ is not geometrically simple}\}$$

are finite.

*Proof.* The abelian variety  $A/K$  has big monodromy. In case i) this follows by Theorem 3.6. In case ii) this is a well-known theorem of Serre, cf. [27], [28]. Define  $K' := F_{\text{sep}}K$ . As  $A/K$  is not weakly isotrivial with respect to  $F$  by assumption, Proposition 4.2 implies that there is a constant  $\ell_0 > \text{char}(K)$  such that  $\mathcal{M}_{K'}(A[\ell]) = \text{Sp}(A[\ell], e_\ell^\lambda)$  for all primes  $\ell \geq \ell_0$ . Hence  $A_{K'}/K'$  has big monodromy. Now Propositions 4 and 7 of [5] imply the assertion. Note that the notion of “big monodromy” in the paper [5] is slightly different from ours.  $\square$

## 5 Appendix. Proof of Theorem 3.4

The aim of this Appendix is to provide a selfcontained proof of Theorem 3.4, which was first proven in the papers [14] and [15]. We have also taken advantage of the exposition in [18].

Let  $\ell > 2$  be a prime number, let  $(V, e)$  be a finite-dimensional symplectic space over  $\mathbb{F}_\ell$  and  $\Gamma = \text{GSp}(V, e)$ . In what follows  $M$  will be a subgroup of  $\Gamma$  which contains a transvection, such that  $V$  is a simple  $\mathbb{F}_\ell[M]$ -module.

**Remark 5.1.** • For a set  $U \subset V$ , we will denote by  $\langle U \rangle$  the vector space generated by  $U$  in  $V$ .

- For a vector  $u \in V$  and a scalar  $\lambda \in \mathbb{F}_\ell$ , we denote by  $T_u[\lambda] \in \Gamma$  the morphism  $v \mapsto v + \lambda e(v, u)u$ . For each transvection  $\tau \in \Gamma$  there exist  $u \neq 0$ ,  $\lambda \neq 0$  such that  $\tau = T_u[\lambda]$ , and  $\langle u \rangle = \ker(\tau - \text{Id})$ . If this is the case we will say that  $\langle u \rangle$  is the direction of  $\tau$ . Each nonzero vector in  $\langle u \rangle$  shall be called a direction vector of  $\tau$ .
- Given a group  $G \subset \Gamma$ , we will denote by  $L(G)$  the set of vectors  $u \in V$  such that there exists a transvection in  $G$  with direction vector  $u$ .
- We will say that a group  $G \subset \Gamma$  fixes a vector space  $W$  if  $\{g(w) : g \in G, w \in W\} \subset W$ .

The proof of Part iii) of Theorem 3.4 is quite simple and is based at the following observation.

**Lemma 5.2.** Let  $G \subseteq \text{GSp}(V)$  be a subgroup and  $R$  the subgroup of  $G$  generated by the transvections in  $G$ . Then for all  $g \in G$ ,  $r \in R$ ,  $grg^{-1} \in R$ .

*Proof.* Note that if  $T = T_v[\lambda] \in G$  is a transvection, then  $gT_v[\lambda]g^{-1} = T_{gv}[\lambda]$  is also a transvection, which belongs to  $G$ , therefore also to  $R$ . Now if we have an element of  $R$ , say  $T_1 \circ \dots \circ T_k$  for certain transvections  $T_1, \dots, T_k$ , then  $g(T_1 \circ \dots \circ T_k)g^{-1} = (gT_1g^{-1}) \circ \dots \circ (gT_kg^{-1})$  is the composition of transvections of  $G$ , therefore an element of  $R$ .  $\square$

Part i) of Theorem 3.4 is essentially Lemma 3.2 of [14]. Before proceeding to prove it, note the following elementary facts.

**Lemma 5.3.** Let  $G$  be a group that acts irreducibly on  $V$ , and let  $W \subset V$  a nonzero vector space. Then  $V = \sum_{g \in G} gW$ .

*Proof.* Let  $S$  be the set  $S = \{g(w) : g \in G, w \in W\}$ . Consider the vector space  $\langle S \rangle$ . This vector space is fixed by  $G$ , hence since  $G$  acts irreducibly on  $V$  it must coincide with  $V$ .  $\square$

**Lemma 5.4.** Let  $W$  be a vector subspace of  $V$ , and assume that it is fixed by a transvection  $T = T_u[\lambda]$ . Then either  $u \in W$  or  $u \in W^\perp$ .

*Proof.* Recall that, for all  $v \in V$ ,  $T(v) = v + \lambda e(v, u)u$ . If  $u \notin W$ , the only way for  $T$  to fix  $W$  is that  $e(w, u) = 0$  for all  $w \in W$ .  $\square$

*Proof of Theorem 3.4, i)*

Consider the action of  $R$  on  $V$ . The first step is to fix one simple nonzero  $R$ -submodule  $W$  contained in  $V$  (This always exists because  $V$  is finite-dimensional as an  $\mathbb{F}_\ell$ -vector space).

By Lemma 5.3, we know that  $V = \sum_{g \in M} gW$ . Moreover, for  $g_1, g_2 \in M$  it holds that  $g_1W = g_2W$  if and only if  $g_1H = g_2H$ . Therefore we can write  $V = \sum_{g \in M/H} gW$ , where  $H$  is the stabilizer of  $W$  in  $M$ . The proof of i) boils down to prove that the sum is direct and orthogonal, that is, if  $g_1H \neq g_2H$ , then  $g_1W \cap g_2W = 0$  and  $g_1W \subset (g_2W)^\perp$ . Equivalently, we will prove that for any  $g \in M$ , if  $gW \neq W$ , then  $gW \cap W = 0$  and  $gW \perp W$ .

The first claim, namely  $gW \neq W$  implies  $gW \cap W = 0$  is easy. The key point is to note that for each  $g \in M$ ,  $gW$  is also fixed by  $R$ . Take  $r \in R$ ,  $gw \in gW$ . Then  $rgw = g(g^{-1}rg)w \in gW$  since  $g^{-1}rg \in R$  by Lemma 5.2 and hence fixes  $W$ . Now it follows that  $W \cap gW$  is fixed by  $R$ , and thus is an  $R$ -subrepresentation of  $W$ . But  $W$  is a simple  $R$ -module, hence since  $W \cap gW \neq W$ , it must follow that  $gW \cap W = 0$ .

To prove that  $gW \neq W$  implies  $gW \perp W$ , we need to make first the following very important observation.

**Claim 5.5.** *The set  $L(M) \cap W$  generates  $W$ .*

*Proof of Claim 5.5.* First let us see that  $L(M) \cap W$  is nontrivial. Since any transvection in  $M$  fixes  $W$  by definition of  $W$ , it follows by Lemma 5.4 that either its direction vector belongs to  $W$ , or else it is orthogonal to  $W$ , in which case the transvection acts trivially on  $W$ . But it cannot happen that all transvections in  $M$  act trivially on  $W$ . For, if a transvection  $T$  acts trivially on  $W$ , then for all  $g \in M$ ,  $gTg^{-1}$  acts trivially on  $gW$ . But since  $R = gRg^{-1}$  (because of Lemma 5.2), then if all  $R$  acts trivially on  $W$ , it also acts trivially on  $gW$ . Now recall that  $V = \sum_{g \in M} gW$ . Then  $R$  would act trivially on  $V$ . But  $R$  contains at least a transvection, and this does not act trivially on  $V$ . We have a contradiction.

Hence  $L(M) \cap W$  is non zero. But now observe that this set is fixed by the action of  $R$ , since the elements of  $M$  bring direction vectors into direction vectors. Therefore the vector space  $\langle L(M) \cap W \rangle \subset W$  is fixed by the action of  $R$ . Since we are assuming  $W$  is a simple  $R$ -module, it follows that  $\langle L(M) \cap W \rangle = W$   $\square$

Now we are able to prove that if  $gW \neq W$ , then  $gW \subset W^\perp$ . Because of the previous claim, it suffices to show that, for any nonzero vector  $w \in W$  which is the direction vector of a transvection in  $M$ , say  $T$ ,  $w \in (gW)^\perp$ . Now recall that, since  $T$  fixes  $gW$ , by Lemma 5.4 either  $w \in gW$  or  $w \in (gW)^\perp$ . But  $gW \cap W = 0$ , so  $w \in (gW)^\perp$ .  $\square$

Before proving Part ii) of Theorem 3.4, we will introduce some notation.

**Definition 5.6.** *Let  $g \in M$ . We will denote by  $R_g$  the subgroup of  $R$  generated*

by the transvections that act non-trivially on  $gW$ .

The following lemma is Lemma 7 of [15].

**Lemma 5.7.** *Let  $g_1, g_2 \in M$  with  $g_1H \neq g_2H$ . Then the commutator  $[R_{g_1}, R_{g_2}]$  is trivial.*

*Proof.* For  $i = 1, 2$ , let  $T_i \in R_{g_i}$  be a transvection. We will see that they commute. By Lemma 5.4 applied to  $g_iW$ , either  $T_i$  acts trivially on  $g_iW$  or its direction vector, say  $u_i$ , belongs to  $g_iW$ . By definition of  $R_{g_i}$  we have the second possibility. But because of Part i) of Theorem 3.4, for each  $g \in M$  such that  $g_iW \neq gW$ ,  $g_iW \cap gW = 0$ , hence  $u_i \notin gW$ . Therefore again by Lemma 5.4 applied now to  $gW$ , it follows that  $T_i$  acts trivially on  $gW$ . Therefore  $T_1$  and  $T_2$  commute on each  $gW$ , since at least one of them acts trivially on it. Since  $V = \bigoplus_{g \in M/H} gW$ , it follows that they commute on all  $V$ .  $\square$

*Proof of Theorem 3.4, ii).*

Let  $M/H = \{g_1H, \dots, g_sH\}$ , with  $g_1 = \text{Id}$ . Define the map

$$P : \prod_{i=1}^s R_{g_i} \rightarrow R$$

$$(r_1, r_2, \dots, r_s) \mapsto r_1 \cdot r_2 \cdots r_s.$$

Since by Lemma 5.7 elements from the different  $R_{g_i}$  commute, this map is a group homomorphism. Let us see that it is also an isomorphism.

Assume that  $r_1 \cdot r_2 \cdots r_s = \text{Id}$ , and that there is a certain  $r_j$  which is not the identity matrix. Then  $r_j$  must act nontrivially on a certain vector  $v \in V$ . Since the elements of  $R_{g_j}$  act trivially on the elements of  $g_iW$  for  $i \neq j$  and  $V = \bigoplus_{i=1}^s g_iW$ , we can assume that  $v \in g_jW$ . But then the remaining  $r_i$  with  $i \neq j$  act trivially on  $v$  and on  $r_j(v)$ . Therefore  $\text{Id}(v) = r_1 \cdots r_s(v) = r_j(v) \neq v$ , which is a contradiction. To prove surjectivity, it suffices to note that each transvection  $T$  of  $M$  belongs to one of the  $R_{g_i}$ , (hence each element of  $R$  can be generated by elements of  $\cup_i R_{g_i}$ ). And this holds because, since  $T$  fixes all the  $g_iW$ , the direction vector of  $T$  must either belong to  $g_iW$  or be orthogonal to it because of Lemma 5.4, and since  $V = \bigoplus_{i=1}^s g_iW$  it cannot be orthogonal to all the  $g_iW$ . Therefore we get that  $R \simeq \prod_{i=1}^s R_{g_i}$ .

Now we are going to apply the following result [33, Main Theorem]:

**Theorem 5.8.** *Suppose  $G \subset \text{GL}(n, k)$  is an irreducible group generated by transvections. Suppose also that  $k$  is a finite field of characteristic  $\ell > 2$ , and that  $n > 2$ . Then  $G$  is conjugate in  $\text{GL}(n, k)$  to one of the groups  $\text{SL}(n, k_0)$ ,  $\text{Sp}(n, k_0)$  or  $\text{SU}(n, k_0)$ , where  $k_0$  is a subfield of  $k$ .*

Note that, if  $n = 2$ , the result is also true and well known (cf. [4, Section 252]).

Now  $R_{g_1}$  is generated by transvections, and acts irreducibly on  $W$  (because  $R$  acts irreducibly on  $W$ , and  $R_{g_1}$  is the group generated by all those transvections

in  $M$  that act nontrivially on  $W$ ). Therefore  $R_{g_1}$  is conjugated to  $\mathrm{Sp}(W)$ . Since all  $R_{g_i}$  are conjugated to  $R_{g_1}$ , the same holds for them. Therefore we have the isomorphism  $R \simeq \prod_{i=1}^s \mathrm{Sp}(W)$ .

Finally, we can view  $H_1 = \prod_{i=1}^s \mathrm{GSp}(W) \simeq \prod_{i=1}^s \mathrm{GSp}(g_i W)$  as the subgroup of  $\Gamma$  fixing each  $g_i W$  and, fixing a symplectic basis on each  $g_i W$ , we can view  $H_2 = \mathrm{Sym}(M/H)$  as the subgroup of  $\Gamma$  that permutes the  $g_i W$  by bringing the fixed symplectic basis of each  $g_i W$  into the fixed symplectic basis of another  $g_j W$ . The group generated by  $H_1$  and  $H_2$  inside  $\Gamma$ , which is the group of elements of  $\Gamma$  that permute the  $g_i W$ , is the semidirect product  $H_1 \rtimes H_2$ .

Recall that  $N_\Gamma(R) = \{g \in \Gamma : gRg^{-1} = R\}$ . Note that  $g \in N_\Gamma(R)$  if and only if for all transvections  $T \in M$ ,  $gTg^{-1} \in R$ . Now, if  $T = T_v[\lambda]$ , it holds that  $gTg^{-1} = T_{g(v)}[\lambda]$ , and this transvection belongs to  $R$  if and only if it is a transvection of  $M$ , that is to say, if and only if  $g(v) \in L(M)$ . Therefore  $g \in N_\Gamma(R)$  if and only if  $g(L(M)) = L(M)$ . Now since  $R$  is isomorphic to  $\prod_{i=1}^s \mathrm{Sp}(g_i W)$ ,  $L(M)$  is the disjoint union of the  $g_i W$ . And moreover, if  $W$  is an  $R$ -module and  $g \in N_\Gamma(R)$ , then  $R$  fixes  $gW$ . Therefore, if  $W$  is a simple  $R$ -module, then  $gW \neq W$  implies that  $gW \cap W = 0$ . Thus if  $g \in N_\Gamma(R)$ , then  $g$  permutes the  $g_i W$ . In other words,  $N_\Gamma(R) \subset \prod_{i=1}^s \mathrm{GSp}(W) \rtimes \mathrm{Sym}(M/H)$ . Reciprocally, each element of  $\prod_{i=1}^s \mathrm{GSp}(W) \rtimes \mathrm{Sym}(M/H)$  carries elements of  $\bigcup_i g_i W$  in elements of  $\bigcup_i g_i W$ , that is to say, carries  $L(M)$  into  $L(M)$ , and therefore belongs to  $N_\Gamma(R)$ .  $\square$

This completes the proof of Part a) of Theorem 3.4.

*Proof of Part b) of Theorem 3.4.* Recall that  $(V, e)$  is a symplectic space over  $\mathbb{F}_\ell$  and  $M$  a subgroup of  $\Gamma := \mathrm{GSp}(V, e)$ .  $M$  contains a transvection and  $V$  is a simple  $\mathbb{F}_\ell[M]$ -module by assumption. Furthermore  $R$  is the subgroup of  $M$  generated by the transvections in  $M$ ,  $0 \neq W \subset V$  is a simple  $\mathbb{F}_\ell[R]$ -module and  $H = \mathrm{Stab}_M(W)$ . We already proved that there is a orthogonal direct sum decomposition  $V = \bigoplus_{g \in M/H} gW$ . Furthermore  $R \cong \prod_{g \in M/H} \mathrm{Sp}(W)$ ,  $N_\Gamma(R) \cong \prod_{g \in M/H} \mathrm{GSp}(W) \rtimes \mathrm{Sym}(M/H)$  and  $R \subset M \subset N_\Gamma(R)$ . Denote by  $\varphi : N_\Gamma(R) \rightarrow \mathrm{Sym}(M/H)$  the projection.

Let  $E/\mathbb{F}_\ell$  be a finite extension and  $\rho : E^\times \rightarrow M \subset \mathrm{GL}(V)$  a representation of amplitude  $\mathrm{amp}(\rho) \leq e$ . Assume that  $\ell > e \dim(V) + 1$ . We have to prove that  $\varphi(\rho(E^\times)) = \{1\}$ .

Define  $S := \ker(\varphi \circ \rho) \subset E^\times$ . Then  $[E^\times : S] \leq |M/H| \leq \dim(V)$ , and this implies  $e[E^\times : S] < \ell - 1$ . Furthermore

$$\rho(S) \subset \ker(\varphi) \cong \prod_{g \in M/H} \mathrm{GSp}(gW).$$

Obviously  $\rho(S)$  commutes with the center

$$Z(\ker(\rho)) \cong \prod_{g \in M/H} \mathbb{F}_\ell^\times \mathrm{Id}_{gW}$$

of  $\ker(\rho)$ . Now by [15, Lemma 3]  $\rho(E^\times)$  commutes with  $Z(\ker(\rho))$ , because  $e[E^\times : S] < \ell - 1$ . It can easily be seen that the centralizer of  $Z(\ker(\rho))$  in

$N_\Gamma(R)$  is equal to  $\ker(\varphi) \cong \prod_{g \in M/H} \mathrm{GSp}(gW)$ . Hence  $\rho(E^\times) \subset \ker(\varphi)$  and this implies  $\varphi \circ \rho(E^\times) = \{1\}$ .  $\square$

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