REMARKS ON A THEOREM OF PINK IN PRESENCE OF BAD REDUCTION

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ABSTRACT. In this note we prove new cases of the Mumford-Tate conjecture by extending a theorem of Richard Pink for abelian varieties without nontrivial endomorphisms and with bad semistable reduction. We use quadratic pairs introduced by J.G.Thompson in the seventies, an important tool in the program of classifying all simple finite groups. Proof of our main result applies the classification of the quadratic pairs as described by Premet and Suprunenko. Along the way we reprove and generalize a theorem of Chris Hall on the image of Tate module representation of abelian variety as above, to all possible values of its toric dimension.

1. INTRODUCTION

Let A be a simple abelian variety of dimension g defined over a number field K. Suppose that A/K has bad (split) semistable reduction at some place v of the field K, denote $\mathbb{F}_v$ the finite residual field. Let $\mathcal{A} \to \mathrm{spec}(\mathcal{O}_K)$ be the Néron model; the connected component of its special fibre lies in an exact sequence of algebraic $\mathbf{F}_v$-groups:

$$(1) \qquad 0 \longrightarrow T = \mathbb{G}_m^s \longrightarrow \mathcal{A}_v^0 \longrightarrow B_v \longrightarrow 0,$$

where B_v is an abelian variety over $\mathbb{F}_v$. We call $s = \dim T$ *the toric dimension* of A at v.

The Mumford-Tate group $\mathrm{MT}(A)$ is an algebraic $\mathbb{Q}$-group (cf. for example [Pi98], for the definition) defined in terms of the complex structure of $A(\mathbb{C})$, that comes together with a representation on $V := H^1(A(\mathbb{C}), \mathbb{Q})$. When the abelian variety is of type I, II or III in Albert classification, the Hodge group $\mathrm{Hdg}(A)$ is the derived group of $\mathrm{MT}(A)$ and we have $\mathrm{MT}(A) = \mathbb{G}_m \mathrm{Hdg}(A)$. Denote $T_\ell(A)$ the Tate module and $V_\ell(A) := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$; the central object of our study is the Galois representation $\rho_\ell : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}(V_\ell(A))$. We denote G_ℓ the connected component of the Zariski closure of $\rho_\ell(\mathrm{Gal}(\bar{K}/K))$ in $\mathrm{GL}(V_\ell(A)) \cong \mathrm{GL}_{2g, \mathbb{Q}_\ell}$, it is an algebraic $\mathbb{Q}_\ell$-group and we still denote ρ_ℓ its representation on V_ℓ. Via the comparison isomorphism $V_\ell \cong V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ we can compare $\mathrm{MT}(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ and G_ℓ inside $\mathrm{GL}_{2g, \mathbb{Q}_\ell}$. A deep result due to Borovoï [Bor74], Deligne [Del82] and Piatetski-Shapiro [Pia] shows that G_ℓ is an algebraic subgroup of $\mathrm{MT}(A) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, the Mumford-Tate conjecture asserts the equality of these two algebraic groups. We refer the reader to a variety of surveys on the current state of arts of the Mumford-Tate conjecture, which is available online.

It is known that the Mumford-Tate conjecture holds for abelian varieties over number fields, if and only if, it holds for simple ones cf. [Com19]. Further, it is easy to see that

replacing K by a finite extension or A by an isogenous abelian variety does not alter the validity of Mumford-Tate conjecture; it is a deeper result (cf. [LaPi95], Theorem 4.3) that Mumford-Tate conjecture does not depend on the chosen prime ℓ. Hence, without loosing any generality, we can concentrate on simple abelian varieties over K, assume semistability and pick any convenient ℓ.

Main Theorem. *Let A be an abelian variety defined over a number field K. Suppose that* $\mathrm{End}(A) = \mathbb{Z}$ *and there is a place of bad semistable reduction with toric dimension s. Then* $\mathrm{MT}(A)_{\mathbb{Q}_\ell} = \mathrm{GSp}_{2g,\mathbb{Q}_\ell} = G_\ell$, *and the Mumford-Tate conjecture holds for A, except may be in following exceptional cases.*

   (1) *There is an odd integer $r \geq 3$ such that $g = \frac{1}{2}\binom{2r}{r}$ and $s = \binom{2r-2}{r-1}$.*
   (2) *There is an integer $t \geq 4$ with $t \equiv 0$ or $1 \mod 4$, such that $g = 2^t$ and $s = g$ or $g/2$.*

We remark that the hypotheses in the theorem are *generic* in the sense that a generic abelian variety of dimension g is simple with endomorphism ring $\mathbb{Z}$, whereas abelian varieties with some bad reduction are far more numerous that abelian varieties with (potential) good reduction everywhere, the typical case being for $g = 1$ where integers are rarer than rational numbers. The first instances of the first exceptional case in Main Theorem occur for $(g, s) = (10, 6), (84, 70), (1716, 924)$ etc.The first instances of the second exceptional case occur for $(g, s) = (16, 8), (16, 16), (32, 16), (32, 32), (256, 128), (256, 256)$ etc.

Comparison with former results

Compared to former attemps at the conjecture (which were often based on computation of Lie algebras of algebraic groups, e.g., [BGK03, BGK06, BGK10]) our method is different. The novelty of this note approach depends on applying theory of representations of finite groups of Lie type directly to comparison of special fibres of two Chevalley group schemes in question, and then lifting equality to their $\mathbb{Q}_\ell$−points (cf. Lemma 2.4) in order to confirm the conjecture. Our main theorem shall be compared to a result of Pink (under the assumption $\mathrm{End}(A)=\mathbb{Z}$) and to the following results of Noot, and Chris Hall.

Theorem 1.1. *(Pink [Pi98], Theorem 5.14) Let A be an abelian variety of dimension g with endomorphism ring $\mathbb{Z}$. Assume that g is neither*

    • *half of a kth power for any odd $k > 1$, nor*
    • *of the shape $\frac{1}{2}\binom{2m}{m}$ for any odd $m \geq 3$.*

Then $G_\ell = \mathrm{GSp}_{2g,\mathbb{Q}_\ell}$ and the Mumford-Tate conjecture holds for A.

Theorem 1.2. *(Noot [No00]) Let A be an abelian variety of dimension 4 with endomorphism ring $\mathbb{Z}$. Assume that A is defined over a number field and has a place with bad semistable reduction. Then $G_\ell = \mathrm{GSp}_{8,\mathbb{Q}_\ell}$ and the Mumford-Tate conjecture holds for A.*

Theorem 1.3. *(C. Hall [Ha11]) Let A be an abelian variety of dimension g with endomorphism ring $\mathbb{Z}$. Assume that A has a place of bad semistable reduction with toric dimension $s = 1$. Then $G_\ell = \mathrm{GSp}_{2g,\mathbb{Q}_\ell}$ and the Mumford-Tate conjecture holds for A.*

Our proof of Main Theorem uses quadratic pairs introduced by J.G.Thompson in [Th71]. We apply the classification of the quadratic pairs described by Premet and Suprunenko in [PS83]. Note that along the way we reprove and generalize Theorem 1.3 to all possible values of s.

2. Quadratic pairs and minuscule representations

Definition 2.1. A *quadratic pair* (G, ρ) is formed by a group G, a faithful irreducible representation $\rho : G \to \mathrm{GL}(V)$ such that the subset $\mathcal{Q} := \{g \in G \setminus \{1\} \mid (\rho(g) - 1_V)^2 = 0\}$ is non empty and generates G. (This is Thompson's definition [Th71] when we assume G finite and V a $\mathbb{F}_p$-vector space; otherwise we understand "generates" as "topologically generates"). A quadratic pair is *polarised* (orthogonal or symplectic) if G leaves invariant a symmetric or antisymmetric non degenerate bilinear form. A quadratic pair is *simple* if G is equal to its commutator group and the quotient of G by its centre is simple.

The main properties of quadratic pairs are a decomposition theorem into (almost) simple blocs and a classification of these simple blocs which are due to Thompson [Th71], see also [Ho76, Sa76] and complemented by Premet-Suprunenko in [PS83].

Proposition 2.2. *Let (G, ρ) be a quadratic pair with V an $\mathbb{F}_p$-vector space.*

(1) *(Thompson's central product theorem) There exist simple quadratic pairs (G_i, ρ_i) such that $G = G_1 \cdots G_t$, and (G, ρ) is isomorphic to $\rho_1 \otimes \cdots \otimes \rho_t : G_1 \cdots G_t \to \mathrm{GL}(V_1 \otimes \ldots \otimes V_t)$. Further, if the pair (G, ρ) is polarised, then each (G_i, ρ_i) is polarised.*

(2) *(Thompson, Premet-Suprunenko) A simple quadratic pair belongs to the following list which is given indicating the root system associated to a Chevalley group and a short (generally accepted) name for the representation.*
- $(A_n, \wedge^j Std)$
- (B_n, Std) *or* $(B_n, Spin)$
- $(C_n, \mathcal{W}_j)$ *with a subrepresentation* $\mathcal{W}_j \subseteq \wedge^j Std$
- (D_n, Std) *or* $(D_n, Spin^{\pm})$
- *Explicitly given finite set of representations of exceptional groups of type E_6, E_7, F_4 and G_2.*

We will require the following elementary lemma. An element $g \in \mathrm{End}(V)$ is *k-unipotent* if $(g - 1)^k = 0$ but $(g - 1)^{k-1} \neq 0$; for example a quadratic element is, by definition, 2-unipotent.

Lemma 2.3. *Let g_1 (resp. g_2) be a k_1-unipotent element in $\mathrm{End}(V_1)$ (resp. a k_2-unipotent element in $\mathrm{End}(V_2)$), then $g_1 \otimes g_2$ is a $(k_1 + k_2 - 1)$-unipotent element in $\mathrm{End}(V_1 \otimes V_2)$. In particular, quadratic elements of a quadratic pair $(G_1 \cdots G_t, \rho_1 \otimes \cdots \otimes \rho_t)$ lie in some G_i.*

Proof. See [Pa04], Lemma 2.2.1. $\square$

Our typical example of quadratic pair is obtained by considering the Galois representation on $A[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^{2g}$ and restricting it to the subgroup R generated by inertia and

their conjugates. One sees that this representation is still irreducible for ℓ large enough. Minuscule representation appear when considering the ℓ-adic representation.

The link between quadratic pairs and minuscule representations is suggested in [Se79] (see the remark after Corollaire, page 180), where it is observed that root elements of a (semi-simple) group act quadratically. To pass from representations mod ℓ to representations over $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$ we'll use the following elementary lemma.

Lemma 2.4. *Let G be a smooth algebraic subgroup of GL_N over $\mathbb{Z}_\ell$ and H a closed subgroup of $G(\mathbb{Z}_\ell)$ such that $\pi(H) = G(\mathbb{F}_\ell)$, where π is the reduction mod ℓ map. Assume $\ell \geq 5$ and the Lie algebra of $G_{\mathbb{F}_\ell}$ is spanned by quadratic elements. Then $H = G(\mathbb{Z}_\ell)$.*

Proof. This is a special case of Lemme 2.6 in [HiRa16]. □

The lemma applies to SL_m, to Sp_{2m} *loc.cit.* and more generally to any quadratic representation ! Thus the property of being spanned by null square matrices is one of the characterisations of minuscule representation, see for example [Pa04] (definition 2.1.3), [Se79] (Lemme 5 and Corollaire after Proposition 5).

Below we recall the table of minuscule weights of height one (reference: [Se79]) giving the type of root system (A,B,C,D), the label of the highest weight describing the representation (labeled as in *loc. cit.*), a short name for the representation, the dimension of the representation and finally the "sign" of the representation: $+1$ for an orthogonal representation, -1 for a symplectic representation, and 0 for a non auto-dual representation.

<div align="center">Table of minuscule representations</div>

Lie type	weight	representation	dimension	sign
A_n $(n \geq 1)$	ϖ_j $(1 \leq j \leq n)$	$\Lambda^j(\mathrm{Std})$	$\binom{n+1}{j}$	$(-1)^j$ if $n = 2j-1$ 0 else
B_n $(n \geq 2)$	ϖ_n	Spin	2^n	$+1$ if $n \equiv 0,3 \mod 4$ -1 if $\equiv 1,2 \mod 4$
C_n $(n \geq 2)$	ϖ_1	Std	$2n$	-1
D_n $(n \geq 3)$	ϖ_1	Std	$2n$	$+1$
	ϖ_{n-1} ϖ_n	Spin$^-$ Spin$^+$	2^{n-1} 2^{n-1}	$\begin{cases} +1 & \text{if } n \equiv 0 \mod 4 \\ -1 & \text{if } n \equiv 2 \mod 4 \\ 0 & \text{if } n \text{ odd} \end{cases}$

3. PROOF OF MAIN THEOREM

Existence of a place of bad reduction with toric dimension s implies that a topological generator of (the image of) inertia provides a *quadratic* element g with *drop* equal to s. Recall that a quadratic element of (G, ρ) is an element such that $(\rho(g) - 1)^2 = 0$ and the drop is the dimension of the image of $\rho(g) - 1$. Let us denote by $G'_\ell/\mathbb{Q}_\ell$ the derived

group of the algebraic monodromy group $G_\ell := \overline{\rho_\ell(\mathrm{Gal}(\bar{K}/K))}^{\mathrm{Zar}}$ which, without loss of generality, can be assumed connected. For simplicity we treat the case $\mathrm{End}(A)=\mathbb{Z}$ although our methods apply to more general situations (cf. Section 4). The algebraic group G'_ℓ and the natural representation $\rho_\ell : G'_\ell \to \mathrm{GL}(T_\ell(A)) \subset \mathrm{GL}_{2g,\mathbb{Q}_\ell}$ have very special features.

- The representation is faithful symplectic.
- (Faltings, [Fa83]) The ℓ-adic representation is absolutely irreducible; further, for ℓ large enough, the representation mod ℓ is still irreducible.
- (Pink, [Pi98]) The representation is a tensor product of representations $\rho_i : G_i \to \mathrm{GL}(V_i)$, where G_i are absolutely irreducible groups, $G'_\ell = G_1 \cdots G_s$ (an almost direct product), $V_\ell \cong V_1 \otimes \cdots \otimes V_s$ and each of these factors is a classical minuscule symplectic representation. Further, the decomposition is isotypical with a transitive action of the Galois group.

We distinguish two cases, using quadratic pairs in the first case and minuscule representations for the second:

- (A) $s > 1$ (hence odd and ≥ 3).
- (B) $s = 1$ (i.e. the group G'_ℓ is absolutely irreducible).

Case (A) Assumption on s implies (denoting $m= \dim V_i$) that $2g=m^s$ as in [Pi98], but we will show that, when there is a place of bad reduction, this case does not occur. We choose ℓ large enough so that the representation modulo ℓ, which we denote $\bar{\rho}_\ell : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}(A[\ell]) \cong \mathrm{GL}_{2g}(\mathbb{F}_\ell)$, is still irreducible. Let $\bar{R}$ be the subgroup generated by the image of the inertia group at the place of bad reduction and its conjugates. We first claim that the restriction of $\bar{\rho}_\ell$ to $\bar{R}$ is still irreducible. Indeed by Clifford's classical theorem, the representation is induced from an irreducible subrepresentation W, but $(\bar{R}, W)$ forms a quadratic pair and then $\bar{R}$ is a finite group of Lie type and has no normal subgroups of small index, at least if ℓ is chosen sufficiently large, therefore $V=W$. Next, $\bar{R}$ is generated by the conjugacy class of a quadratic element, therefore it cannot be an almost direct product $G_1 \cdots G_s$ with $s>1$. The argument for this is that a quadratic element g has to occur in only one factor, i.e., $g=(1,\ldots,g_i,\ldots,1) \in G_1 \cdots G_s$. We now lift $\bar{R}$ to $R \subset G'_\ell$, where R denotes the subgroup generated by inertia in the ℓ-adic representation. Since the Galois group permutes the factors, the inertia cannot sit in only one factor (cf. Lemma 2.3) and we conclude that $s=1$. Note that the case of $\dim A=4$ was initially treated by Noot in Prop. 2.1 and Cor. 2.2, [No00].

Case (B). The list of symplectic minuscule representations of dimension $2g$ of a classical group is short; see the table in previous section; we denote $(G_1, \rho_1) \sim (G_2, \rho_2)$ when the two simple algebraic groups G_i have the same root system and ρ_i correspond to the same weight.

(1) $(G'_\ell, \rho) \sim (\mathrm{Sp}_{2g}, Std)$.
(2) $(G'_\ell, \rho) \sim (\mathrm{SL}_{2r}, \wedge^r Std)$, with odd $r \geq 3$ and $2g = \binom{2r}{r}$.
(3) $(G'_\ell, \rho) \sim (\mathrm{SO}_{2r+1}, Spin)$, with $2g = 2^r$ and $r \equiv 1, 2 \mod 4$.
(4) $(G'_\ell, \rho) \sim (\mathrm{SO}_{2r}, Spin^{\pm})$, with $2g = 2^{r-1}$ and $r \equiv 2 \mod 4$.

In the first case we conclude $G'_\ell = \mathrm{Sp}_{2g,\mathbb{Q}_\ell} = \mathrm{Hdg}(A)_{\mathbb{Q}_\ell}$. In the three last cases we use the computation by Premet-Suprunenko of the drop of a quadratic element (cf. [PS83], Section 3). More precisely, in the second case we see that $\dim A = \frac{1}{2}\binom{2r}{r}$ is very specific. Further, the drop of a non-trivial quadratic element in this representation is $s := \binom{2r-2}{r-1}$ (see [PS83], Lemma 18 and the calculation at p.78, beginning of §2), therefore this case can only happen if the toric dimension at a bad place is equal to s. In the third case we see that $\dim A = 2^{r-1}$ is very specific. Further, the drop of a non-trivial quadratic element in this representation is $s := 2^{r-1}$ or 2^{r-2} (see [PS83] Lemma 20), therefore this case can only happen if the toric dimension at a bad place is equal to $s = g$ or $g/2$. For the spin representation to be symplectic, we need $r \equiv 1, 2 \mod 4$. In the fourth case we see that $\dim A = 2^{r-2}$ is very specific. Further, the drop of a non-trivial quadratic element in this representation is $s := 2^{r-2}$ or 2^{r-3} (see [PS83] Lemma 21 and Note 2, at p.86), therefore this case can only happen, if the toric dimension at a bad place is equal to $s = g$ or $g/2$. For the half-spin representation to be symplectic, we need $r \equiv 2 \mod 4$. This completes the proof of Main Theorem.

Remark 3.1. Our proof shows that inertia subgroups at bad places generate the derived group of the image of Galois. Indeed the above arguments show that the algebraic subgroup R generated by the inertia subgroup and their conjugates is a non trivial normal subgroup and the only such subgroup of Sp_{2g} is Sp_{2g} itself.

4. Conclusion

As a final remark we add the observation that the same method can handle the case when the endomorphism algebra $\mathrm{End}^0(A) := \mathrm{End}(A) \otimes \mathbb{Q}$ is a quaternion algebra with center $\mathbb{Q}$. However, note that the combinatorics, when the center is larger, is much more involved. We plan to address this issue in a future work.

Theorem 4.1. *Let A be an abelian variety defined over a number field K. Suppose that $\mathrm{End}^0(A) := \mathrm{End}(A) \otimes \mathbb{Q}$ is a quaternion algebra with centre $\mathbb{Q}$ and there is a place of bad reduction with toric dimension s (note s is always even). Then,*
- *when the algebra D is indefinite, i.e., A has type II, we have $\mathrm{MT}(A)_{\mathbb{Q}_\ell} = \mathrm{GSp}_{g,\mathbb{Q}_\ell} = G_\ell$ and the Mumford-Tate conjecture holds for A, except may be in following exceptional cases:*
  (1) *there is an odd integer $r \geq 3$ such that $g = \binom{2r}{r}$ and $s = 2\binom{2r-2}{r-1}$,*
  (2) *there is an integer $t \geq 5$ with $t \equiv 1$ or $2 \mod 4$ such that $g = 2^t$ and $s = g$ or $g/2$,*
- *when the algebra D is definite, i.e., A has type III, we have $\mathrm{MT}(A)_{\mathbb{Q}_\ell} = \mathrm{GSO}_{g,\mathbb{Q}_\ell} = G_\ell$ and the Mumford-Tate conjecture holds for A, except may be in following exceptional cases:*
  (1) *there is an even integer $r \geq 2$ such that $g = \binom{2r}{r}$ and $s = 2\binom{2r-2}{r-1}$,*
  (2) *there is an integer $t \geq 4$ with $t \equiv 0$ or $3 \mod 4$ such that $g = 2^t$ and $s = g$ or $g/2$.*

The extension of Theorem 1.1 (and Theorem 4.1) when $s = 2$ to abelian varieties of type II is worked out in [HiRa16], Théorèmes 10.6 and 10.7.

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