J. reine angew. Math., Ahead of Print
DOI 10.1515/crelle-2015-0024

Journal für die reine und angewandte Mathematik

Independence of ℓ-adic representations of geometric Galois groups

By *Gebhard Böckle* at Heidelberg, *Wojciech Gajda* at Poznań and *Sebastian Petersen* at Kassel

Abstract. Let k be an algebraically closed field of arbitrary characteristic, let K/k be a finitely generated field extension and let X be a separated scheme of finite type over K. For each prime ℓ, the absolute Galois group of K acts on the ℓ-adic étale cohomology modules of X. We prove that this family of representations varying over ℓ is almost independent in the sense of Serre, i.e., that the fixed fields inside an algebraic closure of K of the kernels of the representations for all ℓ become linearly disjoint over a finite extension of K. In doing this, we also prove a number of interesting facts on the images and on the ramification of this family of representations.

1. Introduction

Let G be a profinite group and $\mathbb{L}'$ a set of prime numbers; from the middle of Section 4 onward, $\mathbb{L}'$ will denote a certain fixed set of primes. For every $\ell \in \mathbb{L}'$ let G_ℓ be a profinite group and $\rho_\ell\colon G \to G_\ell$ a continuous homomorphism. Denote by

$$\rho\colon G \to \prod_{\ell \in \mathbb{L}'} G_\ell$$

the homomorphism induced by the ρ_ℓ. Following the notation in [35] we call the family $(\rho_\ell)_{\ell\in\mathbb{L}'}$ *independent* if $\rho(G) = \prod_{\ell\in\mathbb{L}'} \rho_\ell(G)$. The family $(\rho_\ell)_{\ell\in\mathbb{L}'}$ is said to be *almost independent* if there exists an open subgroup H of G such that $\rho(H) = \prod_{\ell\in\mathbb{L}'} \rho_\ell(H)$.

The main examples of such families of homomorphisms arise as follows: Let K be a field of characteristic $p \geq 0$ with algebraic closure $\widetilde{K}$ and absolute Galois group

$$\mathrm{Gal}(K) = \mathrm{Aut}(\widetilde{K}/K).$$

Gebhard Böckle is supported by the DFG in the FG 1920 and by the DFG/FNR within the SPP 1489. Wojciech Gajda was supported by the Alexander von Humboldt Foundation and by the National Centre of Sciences of Poland under research grant UMO-2012/07/B/ST1/03541.

Let X/K be a separated algebraic scheme[1)] and denote by $\mathbb{L}$ the set of all prime numbers. For every $q \in \mathbb{N}$ and every $\ell \in \mathbb{L} \smallsetminus \{p\}$ we shall consider the representations

$$\rho_{\ell,X}^{(q)}\colon \mathrm{Gal}(K) \to \mathrm{Aut}_{\mathbb{Q}_\ell}(H^q(X_{\widetilde{K}}, \mathbb{Q}_\ell))$$

and

$$\rho_{\ell,X,c}^{(q)}\colon \mathrm{Gal}(K) \to \mathrm{Aut}_{\mathbb{Q}_\ell}(H_c^q(X_{\widetilde{K}}, \mathbb{Q}_\ell))$$

of $\mathrm{Gal}(K)$ on the étale cohomology groups $H^q(X_{\widetilde{K}}, \mathbb{Q}_\ell)$ and $H_c^q(X_{\widetilde{K}}, \mathbb{Q}_\ell)$. The following independence result has recently been obtained.

Theorem 1.1. *Let K be a finitely generated extension of $\mathbb{Q}$ and let X/K be a separated algebraic scheme. Then the two families $(\rho_{\ell,X}^{(q)})_{\ell\in\mathbb{L}}$ and $(\rho_{\ell,X,c}^{(q)})_{\ell\in\mathbb{L}}$ are almost independent.*

The proof of this statement in the important special case $\mathrm{trdeg}(K/\mathbb{Q}) = 0$ is due to Serre (cf. [35]). The case $\mathrm{trdeg}(K/\mathbb{Q}) > 0$ was worked out in [13], answering a question of Serre (cf. [33, 35]) and of Illusie [20].

The usefulness of almost independence is alluded to in Serre [35, Introduction] (cf. also [33, Section 10]). Almost independence for a family $(\rho_\ell\colon \mathrm{Gal}(K) \to G_\ell)_{\ell\in\mathbb{L}}$ over a field K means that after a finite field extension E/K, the image of $\mathrm{Gal}(E)$ under the product representation $\prod_{\ell\in\mathbb{L}} \rho_\ell$ is the product $\prod_{\ell\in\mathbb{L}} \rho_\ell(\mathrm{Gal}(E))$ of the images. This has applications if one has precise knowledge of the shape of the images for all ℓ. For instance, suppose that there exists a reductive connected algebraic subgroup G of some GL_n over $\mathbb{Q}$ such that, after replacing K by a finite extension K', the image $\rho_\ell(\mathrm{Gal}(K'))$ is open in $G(\mathbb{Q}_\ell) \cap \mathrm{GL}_n(\mathbb{Z}_\ell)$ for all ℓ and surjective for almost all ℓ. Denote by G^{ab} the torus that is the quotient of G by its derived group G^{der}, and assume that the induced family $(\rho_\ell^{\mathrm{ab}}\colon \mathrm{Gal}(K) \to G^{\mathrm{ab}}(\mathbb{Q}_\ell))_{\ell\in\mathbb{L}}$ has adelically open image. Then if G^{der} is simply connected, the almost independence of $(\rho_\ell)_{\ell\in\mathbb{L}}$ implies that the image of $\mathrm{Gal}(K')$ is adelically open, i.e., it is open in the restricted product $\prod'_{\ell\in\mathbb{L}} G(\mathbb{Q}_\ell)$. For the case of general G^{der} we refer the reader to [18], where the authors consider *adelic openness* for geometric families cf. [18, Conjecture 1.1] over number fields. The adelic openness of $(\rho_\ell^{\mathrm{ab}})_{\ell\in\mathbb{L}}$ is in general not a consequence of almost independence. However the case when $(\rho_\ell^{\mathrm{ab}})_{\ell\in\mathbb{L}}$ is a compatibly system of geometric origin is well-understood by [34], and adelic openness holds if K is a number field. The existence of a reductive group G as above, with a priori no condition on G^{der}, is predicted by the Mumford–Tate conjecture (cf. [32, C.3.3, p. 387, C.3.8, p. 389], [33, p. 390]) if $\rho_\ell = \rho_{\ell,X}^{(q)}$ for a smooth projective variety X over a finitely generated extension K of $\mathbb{Q}$.

The present article is concerned with a natural variant of Theorem 1.1 that grew out of the study of independence of families over fields of positive characteristic. For K a finitely generated extension of $\mathbb{F}_p$ it has long been known, e.g., [19] or [11], that the direct analogue of Theorem 1.1 is false: If $\varepsilon_\ell\colon \mathrm{Gal}(\mathbb{F}_p) \to \mathbb{Z}_\ell^\times$ denotes the ℓ-adic cyclotomic character that describes the Galois action on ℓ-power roots of unity, then it is elementary to see that the family $(\varepsilon_\ell)_{\ell\in\mathbb{L}\smallsetminus\{p\}}$ is not almost independent. It follows from this that for every abelian variety A/K, if we denote by $\sigma_{\ell,A}\colon \mathrm{Gal}(K) \to \mathrm{Aut}_{\mathbb{Q}_\ell}(T_\ell(A))$ the representation of $\mathrm{Gal}(K)$ on the ℓ-adic Tate module of A, then $(\sigma_{\ell,A})_{\ell\in\mathbb{L}\smallsetminus\{p\}}$ is *not* almost independent. One is thus led to study

[1)] A scheme X/K is algebraic if the structure morphism $X \to \mathrm{Spec}\, K$ is of finite type (cf. [14, Definition 6.4.1]).

independence over the compositum $\widetilde{\mathbb{F}}_p K$ obtained from the field K by adjoining all roots of unity. Having gone that far, it is then natural to study independence over any field K that is finitely generated over an arbitrary algebraically closed field k. Our main result is the following independence theorem.

Theorem 1.2 (cf. Theorem 7.7). *Let k be an algebraically closed field of characteristic $p \geq 0$. Let K/k be a finitely generated extension and let X/K be a separated algebraic scheme. Then the families $(\rho_{\ell,X}^{(q)}|_{\mathrm{Gal}(K)})_{\ell \in \mathbb{L} \smallsetminus \{p\}}$ and $(\rho_{\ell,X,c}^{(q)}|_{\mathrm{Gal}(K)}))_{\ell \in \mathbb{L} \smallsetminus \{p\}}$ are almost independent.*

It will be clear that many techniques of the present article rely on [35]. Also, some of the key results of [13] will be important. The new methods in comparison with the previous results are the following.

(i) The analysis of the target of our Galois representations, reductive algebraic groups over $\mathbb{Q}_\ell$, will be based on a structural result by Larsen and Pink (cf. [25]) and no longer as for instance in [35] on extensions of results by Nori (cf. [29]). In the proof of Theorem 1.2 we use crucially that there exists a finitely generated subfield K_0 of K and a separated algebraic scheme X_0/K_0 such that $kK_0 = K$ and $X_0 \times_{K_0} \mathrm{Spec}(K) = X$. The group theoretical results mentioned above facilitate greatly the passage from $\mathrm{Gal}(K_0)$ to $\mathrm{Gal}(K)$ when studying their image under $\rho_{\ell,X,?}^{(q)}$.

(ii) Since we also deal with cases of positive characteristic, ramification properties will play a crucial role to obtain necessary finiteness properties of fundamental groups. The results on alterations by de Jong (cf. [6]) will obviously be needed. However we were unable to deduce all needed results from there, despite some known semistability results that follow from [6]. Instead we carry out a reduction to the case where K is absolutely finitely generated and where X/K is smooth and projective (this uses again [6]).

(iii) In the latter case, we use a result by Kerz–Schmidt–Wiesend (cf. [23]) that allows one to control ramification on X by controlling it on all smooth curves on X. By Deligne's results on the Weil conjectures, the semisimplifications of the $\rho_{\ell,X}^{(q)}$ form a pure and strictly compatible system. On curves, we can then apply an ℓ-independence result on tameness from [7] again due to Deligne. Together this allows us to obtain a very clean result on a kind of semistable ramification of $(\rho_{\ell,X}^{(q)})_{\ell \in \mathbb{L} \smallsetminus \{p\}}$, cf. Remark 6.4.

Part (i) is carried out in Section 3. Results on fundamental groups and first results on ramification are the theme of Section 4; there we carry out parts of (ii) and we refine some results from [23]. Section 5 provides the basic independence criterion on which our proof of Theorem 1.2 ultimately rests. For this we introduce notions that describe ramification and semistability in families $(\rho_\ell)_{\ell \in \mathbb{L}}$. Section 6 establishes a semistability property for the families $(\rho_{\ell,X}^{(q)})_{\ell \in \mathbb{L}}$, for any smooth projective variety X over any field K that is finitely generated over a perfect field of positive characteristic. This is step (iii) in the above program. Finally, in Section 7 we complete part (ii) and we give the proof of Theorem 7.7 which is a slightly refined form of Theorem 1.2.

We would like to point out that an alternative proof of part (ii) of our approach could be based on recent unpublished work by Orgogozo which proves a global semistable reduction theorem (cf. [30, Proposition 2.5.8]). When our paper was complete we were informed by Anna Cadoret that, together with Akio Tamagawa, she has proven our Theorem 1.2 by a different method, cf. [5].

Acknowledgement. Gebhard Böckle thanks the Fields Institute for a research stay in the spring of 2012 during which part of this work was written. He also thanks Adam Mickiewicz University in Poznań for making possible a joint visit of the three authors in the fall of 2012. Wojciech Gajda thanks the Interdisciplinary Center for Scientific Computing (IWR) at Heidelberg University for hospitality during research visits in January 2012 and in January 2014. Sebastian Petersen thanks the Mathematics Department at Adam Mickiewicz University for hospitality and support during several research visits. We thank F. Orgogozo and L. Illusie for interesting correspondence concerning this project. In addition, the authors thank the anonymous referee for a thorough review of the paper and many helpful comments and suggestions, and in particular for pointing us to [7, Théorème 9.8] that replaced earlier arguments involving the global Langlands correspondence proven by L. Lafforgue.

2. Notation

Let G be a profinite group. A *normal series* in G is a sequence

$$G = N_0 \triangleright N_1 \triangleright N_2 \triangleright \cdots \triangleright N_s = \{e\}$$

of closed subgroups such that N_{i-1} is a normal subgroup of N_i for all $1 \le i \le 2$. Throughout this manuscript $\mathbb{L}$ denotes the set of all prime numbers. From Section 4 on we define $\mathbb{L}' = \mathbb{L} \smallsetminus \{p\}$ where $p \ge 0$ is the characteristic of a base field k. For any $\ell \in \mathbb{L}$ we denote by G_ℓ^+ the normal hull of the pro-ℓ Sylow subgroups of G.

For a field K with algebraic closure $\widetilde{K}$, we denote by $K_s \subset \widetilde{K}$ a separable closure. Then $\mathrm{Gal}(K)$ is equivalently defined as $\mathrm{Gal}(K_s/K)$ and as $\mathrm{Aut}(\widetilde{K}/K)$, since any field automorphism of K_s fixing K has a unique extension to $\widetilde{K}$. If E/K is an arbitrary field extension, and if $\widetilde{K}$ is chosen inside $\widetilde{E}$, then there is a natural isomorphism

$$\mathrm{Aut}(\widetilde{K}/\widetilde{K} \cap E) \xrightarrow{\cong} \mathrm{Aut}(\widetilde{K}E/E).$$

Composing its inverse with the natural restriction $\mathrm{Gal}(E) \to \mathrm{Aut}(E\widetilde{K}/E)$ one obtains a canonical map

$$\mathrm{res}_{E/K}\colon \mathrm{Gal}(E) \to \mathrm{Gal}(K).$$

For homomorphism $\rho\colon \mathrm{Gal}(K) \to G$ we denote $\rho \circ \mathrm{res}_{E/K}$ by $\rho|_{\mathrm{Gal}(E)}$. If $E \subset \widetilde{K}$, then $\mathrm{res}_{E/K}$ is injective and we identify $\mathrm{Gal}(E)$ with the subgroup $\mathrm{res}_{E/K}(\mathrm{Gal}(E))$ of $\mathrm{Gal}(K)$.

A K-variety X is a scheme X that is integral separated and algebraic over K. We denote by $K(X)$ its function field. A K-curve shall be a K-variety of dimension 1. Let S be a normal connected scheme with function field K. A separable algebraic extension E/K is said to be *unramified along* S if for every finite extension F/K inside E the normalization of S in F is étale over S. We usually consider S as a scheme equipped with the generic geometric base point $s\colon \mathrm{Spec}(\widetilde{K}) \to S$ and denote by $\pi_1(S) := \pi_1(S, s)$ the étale fundamental group of S. If Ω denotes the maximal extension of K in K_s which is unramified along S, then $\pi_1(S)$ can be identified with the Galois group $\mathrm{Gal}(\Omega/K)$. A continuous homomorphism $\rho\colon \mathrm{Gal}(K) \to H$ is said to be *unramified along* S if the fixed field $K_s^{\ker(\rho)}$ is contained in Ω, i.e., if ρ factors through the quotient $\pi_1(S)$ of $\mathrm{Gal}(K)$. In fact, we shall identify continuous homomorphisms $\pi_1(S) \to H$ with continuous homomorphisms $\mathrm{Gal}(K) \to H$ which are unramified along S. If S is a variety defined over a field k, then by a compactification of S we mean a proper k-variety $\overline{S}$ containing S as an open subscheme.

3. Concepts from group theory

In this section, we prove a structural result for compact profinite subgroups of linear algebraic groups over $\widetilde{\mathbb{Q}}_\ell$ (cf. Theorem 3.6) that will be crucial for the proof of the main theorem of this article. It is a consequence of a variant (cf. Proposition 3.10) of a theorem of Larsen and Pink (cf. [25, Theorem 0.2, p. 1106]). The proof of Proposition 3.10 makes strong use of the results and methods in [25], and in particular does not depend on the classification of finite simple groups.

Definition 3.1. For $c \in \mathbb{N}$ and $\ell \in \mathbb{L}$ we denote by $\Sigma_\ell(c)$ the class of profinite groups M which possess a normal series by open subgroups

$$M \triangleright I \triangleright P \triangleright \{1\} \tag{1}$$

such that M/I is a finite product of finite simple groups of Lie type in characteristic ℓ, the group I/P is finite abelian of order prime to ℓ and index $[I : P] \leq c$, and P is a pro-ℓ group.

We observe that if M lies in $\Sigma_\ell(c)$, then the normal series (1) is uniquely determined by M. In fact, P is then the maximal normal pro-ℓ subgroup of M and I is the maximal normal pro-solvable subgroup of M. In particular, P and I are characteristic subgroups of M. Note also that for any group M in $\Sigma_\ell(c)$, the quotient M/M_ℓ^+ is abelian of order at most c.

Definition 3.2. For $d \in \mathbb{N}$ and $\ell \in \mathbb{L}$ we denote by $\mathrm{Jor}_\ell(d)$ the class of finite groups H which possess a normal abelian subgroup N of order prime to ℓ and of index $[H : N] \leq d$. We define $\mathrm{Jor}(d)$ as the union of the $\mathrm{Jor}_\ell(d)$ over all $\ell \in \mathbb{L}$.

The following lemma records a useful permanence property of groups in the classes $\Sigma_\ell(c)$ and $\mathrm{Jor}_\ell(d)$.

Lemma 3.3. *Fix $c, d \in \mathbb{N}$. Then for any $\ell \in \mathbb{L}$ the following holds:*

(a) *If $H' \triangleleft H$ is a normal subgroup of some $H \in \mathrm{Jor}_\ell(d)$, then H' and H/H' lie in $\mathrm{Jor}_\ell(d)$.*

(b) *If $M' \triangleleft M$ is a closed normal subgroup of some $M \in \Sigma_\ell(c)$, then M' and M/M' lie in $\Sigma_\ell(c)$.*

If M' in part (b) of the lemma was a non-normal closed subgroup of M, then clearly M' need not lie in $\Sigma_\ell(c)$ again.

Proof. We only give the proof of (b), the proof of (a) being similar but simpler. Let M be in $\Sigma_\ell(c)$ and consider a normal series

$$M \triangleright I \triangleright P \triangleright \{1\}$$

as in Definition 3.1. Then $L := M/I$ is isomorphic to a product $L_1 \times \cdots \times L_s$ for certain finite simple groups of Lie type L_i in characteristic ℓ. Suppose M' is a closed normal subgroup of M and define $\overline{M'} = M'I/I$. By Goursat's lemma the groups $\overline{M'}$ and $L/\overline{M'}$ are products of some of the L_i. From this it is straightforward to see that both M' and M/M' lie in $\Sigma_\ell(c)$. □

The following corollary is immediate from Lemma 3.3 (b).

Corollary 3.4. *Fix a constant $c \in \mathbb{N}$. Let G be a profinite group, and for each $\ell \in \mathbb{L}$ let $\rho_\ell \colon G \to G_\ell$ be a homomorphism of profinite groups such that $\rho_\ell(G) \in \Sigma_\ell(c)$ for all $\ell \in \mathbb{L}$. Then for any closed normal subgroup $N \triangleleft G$ one has $\rho_\ell(N) \in \Sigma_\ell(c)$ for all $\ell \in \mathbb{L}$.*

Definition 3.5. A profinite group G is called *n-bounded at ℓ* if there exist closed compact subgroups $G_1 \subset G_2 \subset \mathrm{GL}_n(\widetilde{\mathbb{Q}}_\ell)$ such that G_1 is normal in G_2 and $G \cong G_2/G_1$.

The following is the main result of this section.

Theorem 3.6. *For every $n \in \mathbb{N}$ there exists a constant $J'(n)$ (independent of ℓ) such that the following holds: Any group G_ℓ that is n-bounded at some $\ell \in \mathbb{L}$ lies in a short exact sequence*

$$1 \to M_\ell \to G_\ell \to H_\ell \to 1$$

such that M_ℓ is open normal in G_ℓ and lies in $\Sigma_\ell(2^{n-1})$ and H_ℓ lies in $\mathrm{Jor}_\ell(J'(n))$.

We state an immediate corollary:

Corollary 3.7. *Let $\ell > J'(n)$ and let G_ℓ be a profinite group which is n-bounded at ℓ. With notation as in Theorem 3.6 and in Section 2, G_ℓ^+ is an open normal subgroup of M_ℓ of index at most 2^{n-1}.*

In the remainder of this section we shall give a proof of Theorem 3.6. The content of the following lemma is presumably well known.

Lemma 3.8. *For every $r \in \mathbb{N}$, every algebraically closed field F and every semisimple algebraic group G of rank r the center Z of G satisfies $|Z(F)| \le 2^r$.*

Proof. Lacking a precise reference, we include a proof for the reader's convenience. Observe first that the center Z is a finite (cf. [27, I.6.20, p. 43]) diagonalizable algebraic group. Let T be a maximal torus of G. Denote by $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ the character group of T and by $\Phi \subset X(T)$ the set of roots of G. Then $\mathcal{R} = (X(T) \otimes \mathbb{R}, \Phi)$ is a root system. Let $P = \mathbb{Z}\Phi$ be the root lattice and Q the weight lattice of this root system. Then $P \subset X(T) \subset Q$. The center Z of G is the kernel of the adjoint representation (cf. [27, I.7.12, p. 49]). Hence $Z = \bigcap_{\chi\in\Phi} \ker(\chi)$ and there is an exact sequence

$$0 \to Z \to T \to \prod_{\chi\in\Phi} \mathbb{G}_m$$

where the right hand map is induced by the characters $\chi \colon T \to \mathbb{G}_m$ ($\chi \in \Phi$). We apply the functor $\mathrm{Hom}(-, \mathbb{G}_m)$ and obtain an exact sequence

$$\prod_{\chi\in\Phi} \mathbb{Z} \to X(T) \to \mathrm{Hom}(Z, \mathbb{G}_m) \to 0.$$

The cokernel of the left hand map is $X(T)/P$. Thus $|Z(F)| \le [X(T) : P] \le [Q : P]$.

Furthermore, the root system $\mathcal{R}$ decomposes into a direct sum

$$\mathcal{R} = \bigoplus_{i=1}^{s} (E_i, \Phi_i)$$

of indecomposable root systems $\mathcal{R}_i := (E_i, \Phi_i)$. Let $r_i = \dim(E_i)$ be the rank of $\mathcal{R}_i$. Let P_i be the root lattice and Q_i the weight lattice of $\mathcal{R}_i$. Note that by definition we have $P = \bigoplus_i P_i$ and $Q = \bigoplus_i Q_i$. It follows from the classification of indecomposable root systems that $|Q_i/P_i| \le 2^{r_i}$ (cf. [27, Table 9.2, p. 72]) for all i. Hence $|Z(F)| \le |Q/P| \le 2^{r_1}2^{r_2}\cdots 2^{r_s} = 2^r$ as desired. $\square$

Remark 3.9. The semisimple algebraic group $(\mathrm{SL}_{2,\mathbb{C}})^r$ has rank r and its center $(\mu_2)^r$ has exactly 2^r $\mathbb{C}$-rational points. Hence the bound of Lemma 3.8 cannot be improved.

The following result is an adaption of the main result of [25] by Larsen and Pink.

Proposition 3.10. *For every $n \in \mathbb{N}$, there exists a constant $J'(n)$ such that for every field F of positive characteristic ℓ and every finite subgroup Γ of $\mathrm{GL}_n(F)$, there exist normal subgroups L, M, I and P of Γ forming a normal series*

$$\Gamma \rhd L \rhd M \rhd I \rhd P \rhd \{1\}$$

with the following properties:

(i) $[\Gamma : L] \le J'(n)$.

(ii) *The group L/M is abelian of order prime to ℓ.*

(iii) *The group M/I is a finite product of finite simple groups of Lie type in characteristic ℓ.*

(iv) *The group I/P is abelian of order prime to ℓ and $[I : P] \le 2^{n-1}$.*

(v) *P is an ℓ-group.*

Furthermore, the constant $J'(n)$ is the same as in [25, Theorem 0.2, p. 1106].

Proof. We can assume that the field F is algebraically closed. Let $J'(n)$ be the constant from [25, Theorem 0.2, p. 1106]. Larsen and Pink construct in the proof of their theorem ([25, Theorem 0.2, pp. 1155–1156]) normal subgroups Γ_i of Γ such that there is a normal series

$$\Gamma \rhd \Gamma_1 \rhd \Gamma_2 \rhd \Gamma_3 \rhd \{1\}$$

and such that $[\Gamma : \Gamma_1] \le J'(n)$, Γ_1/Γ_2 is a product of finite simple groups of Lie type in characteristic ℓ, Γ_2/Γ_3 is abelian of order prime to ℓ and Γ_3 is an ℓ-group. The construction of the groups Γ_i in [25, Theorem 0.2, pp. 1155–1156] shows that there exists a smooth algebraic group G over F containing Γ such that, if we denote by R the unipotent radical of the connected component G° of G and by $\overline{Z}$ the center of the reductive group $\overline{G} := G^\circ/R$, then $\Gamma_1 \lhd G^\circ(F)$, $\Gamma_3 = \Gamma \cap R(F)$ and Γ_2/Γ_3 is contained in $\overline{Z}(F)$. Let $\overline{D} = [\overline{G}, \overline{G}]$ be the derived group of $\overline{G}$ and $D = [G^\circ, G^\circ]R$.

Now define $L = \Gamma_1$, $M = \Gamma_1 \cap D(F)$, $I = \Gamma_2 \cap D(F)$ and $P = \Gamma_3$. These groups are normal in Γ, because $D(F)$ is characteristic in $G^\circ(F)$ and because $\Gamma_1, \Gamma_2, \Gamma_3$ are normal in Γ. The group L/M is a subgroup of the abelian group $G^\circ(F)/D(F)$. As G°/D is isomorphic to the torus $\overline{G}/\overline{D}$, it follows that the order of L/M is prime to ℓ. The group M/I is a normal subgroup of Γ_1/Γ_2, hence it is a product of finite simple groups of Lie type in characteristic ℓ. The group I/P is a subgroup of Γ_2/Γ_3, hence I/P is abelian of order prime to ℓ. Furthermore, $I/P = I/\Gamma_3$ is a subgroup of $\overline{G}(F)$ which lies in $\overline{D}(F)$ and in $\overline{Z}(F)$. Thus I/P lies in the center $\overline{Z}(F) \cap \overline{D}(F)$ of the semisimple group $\overline{D}(F)$. It follows by Lemma 3.8 that $[I : P] \le 2^{\mathrm{rk}(\overline{D})}$.

It remains to show that $\mathrm{rk}(\overline{D}) \le n - 1$. Let T be a maximal torus of $\overline{D}$ and denote by $\pi : G^\circ \to \overline{G}$ the canonical projection. Note that π induces an epimorphism $[G^\circ, G^\circ] \to \overline{D}$. The algebraic group $B := \pi^{-1}(T) \cap [G^\circ, G^\circ]$ sits in an exact sequence

$$0 \to R \cap [G^\circ, G^\circ] \to B \to T \to 0$$

and B is connected smooth and solvable, because R and T have these properties. The above exact sequence splits (cf. [10, XVII.5.1]); hence B contains a copy of T. This copy is contained in a maximal torus T' of $\mathrm{SL}_{n,F}$ because B is a subgroup of $\mathrm{SL}_{n,F}$. Thus

$$n - 1 = \dim(T') \ge \dim(T) = \mathrm{rk}(\overline{D})$$

as desired. □

Proof of Theorem 3.6. Suppose the group G_ℓ is n-bounded at ℓ, so that it is a quotient $G_{2,\ell}/G_{1,\ell}$ with $G_{i,\ell} \subset \mathrm{GL}_n(\widetilde{\mathbb{Q}}_\ell)$. By Lemma 3.3, it will suffice to prove the theorem in the case $G_\ell = G_{2,\ell}$. Thus we assume that G_ℓ is a compact profinite subgroup of $\mathrm{GL}_n(\widetilde{\mathbb{Q}}_\ell)$. By the compactness of G_ℓ and a Baire category type argument (cf. [12, proof of Corollary 5]) the group G_ℓ is contained in $\mathrm{GL}_n(E)$ for some finite extension E of $\mathbb{Q}_\ell$. Let $\mathcal{O}_E$ be the ring of integers of the local field E. Again by compactness of G_ℓ one can then find an $\mathcal{O}_E$-lattice in E^n that is stable under G_ℓ. Hence we may assume that G_ℓ is a closed subgroup of $\mathrm{GL}_n(\mathcal{O}_E)$.

Let $\mathfrak{p}$ be the maximal ideal of the local ring $\mathcal{O}_E$ and let $\mathbb{F} = \mathcal{O}_E/\mathfrak{p}$ be its residue field. The kernel $\mathcal{K}$ of the canonical map $p : \mathrm{GL}_n(\mathcal{O}_E) \to \mathrm{GL}_n(\mathbb{F})$ is a pro-ℓ group. Hence $Q_\ell = \mathcal{K} \cap G_\ell$ is pro-ℓ and open normal in G_ℓ. We now apply Proposition 3.10 to the finite subgroup G_ℓ/Q_ℓ of $\mathrm{GL}_n(\mathbb{F}) \subset \mathrm{GL}_n(F)$ with $F = \widetilde{\mathbb{F}} \cong \widetilde{\mathbb{F}}_\ell$. This yields normal subgroups L_ℓ, M_ℓ, I_ℓ and P_ℓ of G_ℓ such that there is a normal series

$$G_\ell \rhd L_\ell \rhd M_\ell \rhd I_\ell \rhd P_\ell \rhd Q_\ell \rhd \{1\}$$

with the following properties: The group G_ℓ/M_ℓ lies in $\mathrm{Jor}_\ell(J'(n))$, and the group M_ℓ lies in $\Sigma_\ell(2^{n-1})$ – for the latter use that Q_ℓ is pro-ℓ and normal in G_ℓ and P_ℓ/Q_ℓ is a finite ℓ-group. □

4. Fundamental groups: Finiteness properties and ramification

The purpose of this section is to recall some finiteness properties of fundamental groups and to provide some basic results on ramification. Regarding the latter we draw from results by Kerz, Schmidt and Wiesend (cf. [23]).

We begin with a finiteness result of which a key part is from [13].

Proposition 4.1. *Suppose that either k is a finite field and S is a smooth proper k-variety or that k is a number field and S is a smooth k-variety, and denote by $K = k(S)$ the function field of S. For $d \in \mathbb{N}$, let $\mathcal{M}_d$ be the set of all finite Galois extensions E/K inside $\widetilde{K}$ such that* $\mathrm{Gal}(E/K)$ *satisfies* $\mathrm{Jor}(d)$ *and such that E is unramified along S. Then there exists a finite Galois extension K'/K which is unramified along S such that $E \subset \tilde{k}K'$ for every $E \in \mathcal{M}_d$.*

Proof. For every $E \in \mathcal{M}_d$ the group $\mathrm{Gal}(E/K)$ satisfies $\mathrm{Jor}(d)$ and hence there is a finite Galois extension E'/K inside E such that $[E' : K] \leq d$ and such that E/E' is abelian. Consider the composite fields

$$\Omega' = \prod_{E \in \mathcal{M}_d} E' \subset \Omega = \prod_{E \in \mathcal{M}_d} E.$$

Then Ω/Ω' is abelian. Let k_0 (resp. κ', resp. κ) be the algebraic closure of k in K (resp. in Ω', resp. in Ω),

$$\begin{tikzcd}
 & K \arrow[r, dash] \arrow[d, dash] & \Omega' \arrow[r, dash] \arrow[d, dash] & \Omega \arrow[d, dash] \\
k \arrow[r, dash] & k_0 \arrow[r, dash] & \kappa' \arrow[r, dash] & \kappa.
\end{tikzcd}$$

It suffices to prove the following:

Claim. *The extension $\Omega/\kappa K$ is finite.*

In fact, once this is shown, it follows that the finite separable extension $\Omega/\kappa K$ has a primitive element ω. Then $\Omega = \kappa K(\omega)$ and $K(\omega)/K$ is a finite separable extension. Let K' be the normal closure of $K(\omega)/K$ in Ω. Then $\tilde{k}K' \supset \kappa K' \supset \kappa K(\omega) = \Omega$ as desired.

In the case where k is a number field the claim has been shown in [13, Proposition 2.2]. Assume from now on that k is finite. It remains to prove the claim in that case. The structure morphism $S \to \mathrm{Spec}(k)$ of the smooth scheme S factors through $\mathrm{Spec}(k_0)$ and S is a geometrically connected k_0-variety. The profinite group $\pi_1(S \times_{k_0} \mathrm{Spec}(\tilde{k}))$ is topologically finitely generated (cf. [16, Theorem X.2.9]) and $\mathrm{Gal}(k_0) \cong \hat{\mathbb{Z}}$. Thus it follows by the exact sequence (cf. [16, Theorem IX.6.1])

$$1 \to \pi_1(S \times_{k_0} \mathrm{Spec}(\tilde{k})) \to \pi_1(S) \to \mathrm{Gal}(k_0) \to 1$$

that $\pi_1(S)$ is topologically finitely generated. Thus there are only finitely many extensions of K in $\tilde{K}$ of degree $\leq d$ which are unramified along S. It follows that Ω'/K is a *finite* extension. Thus κ' is a finite field. If we denote by S' the normalization of S in Ω', then $S' \to S$ is finite and étale, hence S' is a smooth proper geometrically connected κ'-variety. Furthermore, Ω/Ω' is abelian and unramified along S'. Hence $\Omega/\kappa\Omega'$ is finite by Katz–Lang (cf. [21, Theorem 2, p. 306]). As Ω'/K is finite, it follows that $\Omega/\kappa K$ is finite. □

To introduce below a notion of tameness that is inspired by [23] and applies to coverings of general schemes, we require further notation. For a Galois extension E/K of fields, a discrete valuation $v \colon K^\times \to \mathbb{Z}$ of K and an extension w of v to E we define $I_{E/K}(w)$ (resp. $I_{E/K}(v)$) to be the inertia group of w (resp. of v) in the extension E/K.[2)] Note that $I_{E/K}(v)$ is well-defined only up to conjugation. We put $I(v) = I_{K_s/K}(v)$. In the special case where v is the trivial valuation, the valuation w must be trivial as well and $I_{E/K}(v)$ is the trivial group. Now let p be the residue characteristic of v and let $\Lambda \subset \mathbb{L} \smallsetminus \{p\}$. The extension E/K shall be called Λ*-tame at* v if the order of $I_{E/K}(v)$ (viewed as a supernatural number) is divisible

[2)] If E/K is infinite, then w need no longer be discrete but its restriction to any finite Galois subextension of E/K is so. For any E/K, the group $I_{E/K}(w)$ is the inverse limit over ramification groups of finite extensions.

only by primes in Λ. Note that E/K is $\mathbb{L} \smallsetminus \{p\}$-tame at v if and only if $I_{E/K}(v)$ is a group of order prime to p, i.e., if and only if E/K is tame at v in the usual sense.[3)] For us the case where $\Lambda = \{\ell\}$ for a single prime number $\ell \neq p$ will be particularly important, and in that case we speak of ℓ-tameness rather than of $\{\ell\}$-tameness. If K/k is a finitely generated extension of fields, then we will denote by $V_{K/k}$ the set of all discrete valuations $K^\times \to \mathbb{Z}$ which are trivial on k.

For the rest of this section let k be a field of characteristic $p \geq 0$, $\mathbb{L}' = \mathbb{L} \smallsetminus \{p\}$ and $\Lambda \subset \mathbb{L}'$. Furthermore, let S be a regular variety over k and $K = k(S)$ its function field. Let G be a locally compact topological group and $\rho\colon \pi_1(S) \to G$ a continuous homomorphism. Let E be the fixed field of $\ker(\rho)$ *in K_s.*

Recall that we identify continuous homomorphisms $\rho\colon \pi_1(S) \to G$ with continuous homomorphisms $\rho\colon \mathrm{Gal}(K) \to G$ which are unramified along S.

Definition 4.2. Let $v \in V_{K/k}$. The homomorphism ρ is said to be Λ-*tame at* v if the order of the profinite group $\rho(I(v))$ (viewed as a supernatural number) is divisible only by prime numbers in Λ. The homomorphism ρ is called Λ-*tame* if it is Λ-tame at every $v \in V_{K/k}$.

Note that the homomorphism ρ is Λ-tame at v if and only if the extension E/K is Λ-tame at v.

Lemma 4.3. *Let $v \in V_{K/k}$. Then there exists a normal compactification $\overline{S}$ of S and a codimension* 1 *point $s \in \overline{S}$ such that $v = v_s$ is the discrete valuation of K attached to s.*

Proof. Let $\overline{S}_0$ be a normal compactification of S, which exists by the theorem of Nagata [26]. By [37, Proposition 6.4], there exists a blow-up $\overline{S}$ of $\overline{S}_0$ with center outside S such that v is the valuation of a codimension 1 point $s \in \overline{S}$. By normalization, we may further assume that $\overline{S}$ is normal. Both operations, blow-up and normalization, do not affect S, and so there exist a normal compactification $\overline{S}$ of S that contains a codimension 1 point s with valuation $v = v_s$. □

Remark 4.4. As an immediate consequence of Lemma 4.3 we see that the following statements are equivalent.

(a) The homomorphism ρ is Λ-tame.

(b) For every normal compactification $\overline{S}$ of S and every codimension 1 point $s \in \overline{S}$ the extension E/K is Λ-tame in the discrete valuation v_s of K attached to s.

In particular, ρ is $\mathbb{L}'$-tame if and only if E/K is divisor tame in the sense of [23].

For a morphism $f\colon S' \to S$, we denote by $f_*\colon \pi_1(S') \to \pi_1(S)$ the induced continuous homomorphism of fundamental groups. The following base change property of Λ-tameness is quite useful.

[3)] Note that if the residue field extension at the valuation v for E/K is inseparable, then p will divide the order of $I_{E/K}(v)$.

Lemma 4.5. *Let k'/k be an arbitrary field extension and S' a regular k'-variety. Let $K' = k'(S')$ and recall that $K = k(S)$. Assume that there is a diagram*

$$\begin{array}{ccc} S' & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k') & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

where f is dominant.

(a) *If $\rho\colon \pi_1(S) \to G$ is Λ-tame, then the composite homomorphism*

$$\rho \circ f_*\colon \pi_1(S') \to \pi_1(S) \to G$$

is Λ-tame.

(b) *If f is finite, K'/K is purely inseparable and $\rho \circ f_*$ is Λ-tame, then ρ is Λ-tame.*

Proof. Recall that E is the fixed field of $\ker(\rho)$. Let E' be the fixed field of $\ker(\rho \circ f_*)$. Then $E' = EK'$ in some separable closure $K'_s \supset K_s$ of K', and we have a diagram of fields

$$\begin{array}{ccc} E & \text{——} & E' \\ | & & | \\ K & \text{——} & K' \end{array}$$

where E/K and E'/K' are Galois. Let $v' \in V_{K'/k'}$ and $v = v'|K$. The restriction map

$$r\colon \mathrm{Gal}(E'/K') \to \mathrm{Gal}(E/K), \quad \sigma \mapsto \sigma|E$$

is injective because $E' = EK'$. It is easy to check that $r(I_{E'/K'}(v'))$ is conjugate to a closed subgroup of $I_{E/K}(v)$. If ρ is Λ-tame, then the order of $I_{E/K}(v)$ is divisible only by primes in Λ, and thus the order of $I_{E'/K'}(v')$ is divisible only by prime numbers in Λ; hence E'/K' is then Λ-tame at v' as desired. This proves part (a).

To prove (b) assume that f is finite, that K'/K is purely inseparable and that $\rho \circ f_*$ is Λ-tame. Then E'/K' is Λ-tame at v'. As K'/K is purely inseparable, the map r is an isomorphism and $r(I_{E'/K'}(v'))$ is conjugate $I_{E/K}(v)$. Thus the order of $I_{E/K}(v)$ is divisible only by primes in Λ and it follows that ρ is Λ-tame. This completes the proof of part (b). □

The following proposition is a useful criterion to establish Λ-tameness for a given homomorphism $\pi_1(S) \to G$. It is a variant of parts of [23, Theorem 4.4].

Proposition 4.6. *Assume that for every regular curve C/k and for every morphism $f\colon C \to S$ the homomorphism*

$$\rho \circ f_*\colon \pi_1(C) \to \pi_1(S) \to G$$

is Λ-tame. Then ρ is Λ-tame.

Proof. We can assume that G is finite and ρ is surjective. Let $v \in V_{K/k}$ and let w be an extension of v to E. Let $I = I(w)$ and $J = \rho(I)$. Then J is solvable. We have to prove that the order of J is divisible only by primes in Λ. Assume to the contrary that there exists a prime number $\ell' \in \mathbb{L} \smallsetminus \Lambda$ such that $|J|$ is divisible by ℓ'. Then, by the solvability of J, there exists a subgroup J_1 of J and a normal subgroup J_2 of J_1 such that $J_1/J_2 \cong \mathbb{Z}/\ell'$. For $i \in \{1,2\}$ let K_i be the fixed field of $\rho^{-1}(J_i)$, let S_i be the normalization of S in K_i and w_i the restriction of w to K_i. Then w_2 is totaly ramified in K_2/K_1, and restricting ρ to $\pi_1(S_1)$ yields an epimorphism $\pi_1(S_1) \to J_1 \to \mathbb{Z}/\ell'$. By Lemma 4.3 there exists a normal compactification $\overline{S}_1$ of S_1 such that w_1 is the discrete valuation attached to some codimension 1 point s_1 of $\overline{S}_1$. As $\overline{S}_1$ is regular in codimension 1, it follows that the maximal regular open subscheme W_1 of $\overline{S}_1$ contains s_1. Furthermore, $S_1 \subset W_1$.

Now let C_1/k be an arbitrary regular curve and let $f\colon C_1 \to W_1$ be a non-constant morphism with $f(C_1) \cap S_1 \neq \varnothing$. Let $D_1 = f^{-1}(S_1)$. For every discrete valuation u on $k(C_1)$ the composite homomorphism

$$\rho'\colon \pi_1(D_1) \to \pi_1(S_1) \to J_1/J_2 \cong \mathbb{Z}/\ell'$$

maps the inertia group $I(u)$ to zero, because $\rho'(I(u))$ is of order divisible only by primes in Λ and a subgroup of $\mathbb{Z}/\ell'$ at the same time. In particular, ρ' factors through $\pi_1(C_1)$. This implies that $S_2 \times_{S_1} D_1 \to D_1$ extends to a not necessarily connected étale cover of C_1. Now by [23, Proposition 4.1], which can be paraphrased as *curve-unramifiedness implies unramifiedness over a regular base*, it follows that the normalization W_2 of W_1 in K_2 is étale over W_1. But then K_2/K_1 is étale along w, a contradiction. □

Remark 4.7. Combining notions in [23] with our notion of Λ-tameness, it is straightforward to define a notion of Λ-curve-tameness. Then Proposition 4.6 asserts that Λ-curve-tameness implies Λ-tameness. Following [23] one can show that in fact the two notions are equivalent.

5. An independence criterion

Throughout this section let k be a field of characteristic $p \geq 0$ and $\mathbb{L}' = \mathbb{L} \smallsetminus \{p\}$. Let S/k be a regular k-variety with function field $K = k(S)$. For every $\ell \in \mathbb{L}'$ let G_ℓ be a locally compact topological group and $\rho_\ell\colon \mathrm{Gal}(K) \to G_\ell$ a continuous homomorphism.

If for all $\ell \in \mathbb{L}'$ the groups $\rho_\ell(\mathrm{Gal}(K))$ are n-bounded at ℓ, then by Theorem 3.6 we have a short exact sequence

$$1 \to M_\ell \to \rho_\ell(\mathrm{Gal}(K)) \to H_\ell \to 1$$

with $H_\ell \in \mathrm{Jor}_\ell(J'(n))$ and $M_\ell \in \Sigma_\ell(2^{n-1})$. In this section, we shall show in Proposition 5.5 and Theorem 5.8 how to control H_ℓ and M_ℓ in a uniform manner, if one has a uniform control on ramification. We begin by introducing the necessary concepts and then give the result.

Recall from [6, Section 2.20] that a morphism $f : V \to U$ between k-varieties is an *alteration* if it is proper and surjective and there exists a dense open subscheme U' of U such that $f^{-1}(U') \to U'$ is finite.

Definition 5.1. The family $(\rho_\ell)_{\ell \in \mathbb{L}'}$ satisfies

(i) condition $\mathcal{R}(S/k)$ if there exists a dense open subscheme U of S such that for every $\ell \in \mathbb{L}'$ the homomorphism ρ_ℓ factors through $\pi_1(U)$.

(ii) condition $\mathcal{S}(S/k)$ if there exists a dense open subscheme U of S, a regular k-variety V and an alteration $f: V \to U$ such that for every $\ell \in \mathbb{L}'$ the homomorphism ρ_ℓ factors through $\pi_1(U)$ and such that for every $\ell \in \mathbb{L}'$ the composite homomorphism

$$\rho_\ell \circ f_*: \pi_1(V) \to \pi_1(U) \to G_\ell$$

is ℓ-tame. Such a triple (U, V, f), or simply $f: V \to U$, is called a witness of the condition $\mathcal{S}(S/k)$, or we say that it witnesses the condition $\mathcal{S}(S/k)$.

Note that condition $\mathcal{S}(S/k)$ implies condition $\mathcal{R}(S/k)$. The condition $\mathcal{R}(S/k)$ is a uniform constructibility condition; $\mathcal{S}(S/k)$ is a uniform semistability condition. Example 5.3 shows that both conditions are satisfied for the family of ℓ-adic representations attached to an abelian variety A over the function field K of S.

Lemma 5.2. *Assume that $(\rho_\ell)_{\ell \in \mathbb{L}'}$ satisfies condition $\mathcal{S}(S/k)$.*

(a) *If $f: V \to U$ witnesses condition $\mathcal{S}(S/k)$, then for any alteration $g: W \to V$ with W regular, the composition $f \circ g: W \to U$ witnesses condition $\mathcal{S}(S/k)$.*

(b) *There exists a witness (U', V', f') of condition $\mathcal{S}(S/k)$ such that f' is finite étale.*

Part (b) is useful in Proposition 5.5 when studying the property $\mathcal{S}(S/k)$ under base change: being finite étale is preserved under base change, while being an alteration is not.

Proof. The hypothesis in (a) means that ρ_ℓ factors through $\pi_1(U)$ and the composition $\rho_\ell \circ f_*: \pi_1(V) \to G_\ell$ is ℓ-tame, for any ℓ in $\mathbb{L}'$. Lemma 4.5 now implies that $\rho_\ell \circ f_* \circ g_*$ is ℓ-tame because g is dominant. Since the composition of alterations is an alteration, part (a) holds true.

To prove (b), let f be as in (a) and let E be the maximal separable extension of K inside $k(V)$. Then E/K is separable and $k(V)/E$ is purely inseparable. Let T be the normalization of U in E. There exists a dense open subscheme U' of U such that the restriction of f to a morphism $V' := f^{-1}(U) \to U'$ is finite. By generic smoothness, after shrinking U' further, we can assume that the canonical morphism $f': T' := U' \times_U T \to U'$ is finite and étale. Let $h: V' \to T'$ be the canonical morphism. We know by assumption that the homomorphism

$$\rho_\ell \circ f'_* \circ h_* = \rho_\ell \circ (f|V')_*: \pi_1(V') \to G_\ell$$

is ℓ-tame for all $\ell \in \mathbb{L}'$. Part (b) of Lemma 4.5 now shows that $\rho_\ell \circ f'_*: \pi_1(T') \to G_\ell$ is ℓ-tame for all $\ell \in \mathbb{L}'$. Hence f' witnesses condition $\mathcal{S}(S/k)$ as desired. □

Example 5.3. Let A/K be an abelian variety. For every $\ell \in \mathbb{L}'$ denote by

$$\rho_{\ell,A}: \mathrm{Gal}(K) \to \mathrm{Aut}_{\mathbb{Z}_\ell}(T_\ell(A))$$

the representation of $\mathrm{Gal}(K)$ on the ℓ-adic Tate module

$$T_\ell(A) = \varprojlim_{i \in \mathbb{N}} A[\ell^i].$$

By the spreading-out principles of [15] there exists a non-empty open subscheme U of S and an abelian scheme $\mathcal{A}$ over U with generic fiber A. This implies (cf. [17, IX.2.2.9]) that $\rho_{\ell,A}$ is unramified along U, i.e., that $\rho_{\ell,A}$ factors through $\pi_1(U)$ for every $\ell \in \mathbb{L}'$. Hence the family $(\rho_{\ell,A})_{\ell\in\mathbb{L}'}$ satisfies condition $\mathcal{R}(S/k)$.

In order to obtain also condition $\mathcal{S}(S/k)$ from Definition 5.1, we choose an odd prime $\ell_0 \in \mathbb{L}'$, and we define $K' = K(A[\ell_0])$. After shrinking U accordingly we can assume that the normalization U' of U in K' is étale over U. Now let $v' \in V_{K'/k}$ be a non-trivial discrete valuation and $R_{v'}$ the discrete valuation ring of v'. Let $\mathcal{N}_{v'}/\operatorname{Spec}(R_{v'})$ be the Néron model of A over $R_{v'}$. The condition $K' \supset K(A[\ell_0])$ forces $\mathcal{N}_{v'}$ to be semistable (cf. [17, IX.4.7]). This in turn implies that $\rho_{\ell,A}|I(v')$ is unipotent (and hence $\rho_{\ell,A}(I(v'))$ is pro-ℓ) for *every* $\ell \in \mathbb{L}'$ (cf. [17, IX.3.5]). It follows that the family $(\rho_{\ell,A})_{\ell\in\mathbb{L}'}$ satisfies condition $\mathcal{S}(S/k)$.

Recall that for a separated algebraic scheme X/K, for $q \in \mathbb{N}$ and for every $\ell \in \mathbb{L} \smallsetminus \{p\}$ we consider the representations

$$\rho_{\ell,X}^{(q)}\colon \operatorname{Gal}(K) \to \operatorname{Aut}_{\mathbb{Q}_\ell}(H^q(X_{\widetilde{K}}, \mathbb{Q}_\ell))$$

and

$$\rho_{\ell,X,c}^{(q)}\colon \operatorname{Gal}(K) \to \operatorname{Aut}_{\mathbb{Q}_\ell}(H_c^q(X_{\widetilde{K}}, \mathbb{Q}_\ell))$$

of $\operatorname{Gal}(K)$ on the étale cohomology groups $H^q(X_{\widetilde{K}}, \mathbb{Q}_\ell)$ and $H_c^q(X_{\widetilde{K}}, \mathbb{Q}_\ell)$.

Proposition 5.4. *For a separated algebraic scheme X/K, the two families $(\rho_{\ell,X}^{(q)})_{\ell\in\mathbb{L}'}$ and $(\rho_{\ell,X,c}^{(q)})_{\ell\in\mathbb{L}'}$ both satisfy condition $\mathcal{R}(S/k)$.*

Proof. There exists a separated morphism $f\colon \mathcal{X} \to S$ of finite type with generic fiber X. Next there exists a dense open subscheme U of S such that for every $\ell \in \mathbb{L}'$ the sheaves $R^q f_*(\mathbb{Q}_\ell)|U$ and $R^q f_!(\mathbb{Q}_\ell)|U$ are lisse (cf. [22, Theorems 3.1.2–3.1.3], [20, Corollary 2.6]). Let $\overline{\xi}\colon \operatorname{Spec}(\widetilde{K}) \to U$ be the geometric generic point of U afforded by the choice of $\widetilde{K}$. Then the stalk $R^q f_*(\mathbb{Q}_\ell)_{\overline{\xi}}$ (resp. $R^q f_!(\mathbb{Q}_\ell)_{\overline{\xi}}$) is $H^q(X_{\widetilde{K}}, \mathbb{Q}_\ell)$ (resp. $H_c^q(X_{\widetilde{K}}, \mathbb{Q}_\ell)$), cf. [2, Théorème VIII.5.5.2, p. 386]. Thus the representations $\rho_{\ell,X}^{(q)}$ and $\rho_{\ell,X,c}^{(q)}$ factor through $\pi_1(U)$ for every $\ell \in \mathbb{L}'$. □

For $p > 0$, we shall treat condition $\mathcal{S}(S/k)$ for the families $(\rho_{\ell,X}^{(q)})_{\ell\in\mathbb{L}'}$ and $(\rho_{\ell,X,c}^{(q)})_{\ell\in\mathbb{L}'}$ in Corollary 7.4. Both conditions $\mathcal{R}(S/k)$ and $\mathcal{S}(S/k)$ behave well under base change in the following sense.

Proposition 5.5. *Let k'/k be an arbitrary field extension and S' a regular k'-variety. Assume that there is a diagram*

$$\begin{tikzcd} S' \arrow[r, "f"] \arrow[d] & S \arrow[d] \\ \operatorname{Spec}(k') \arrow[r] & \operatorname{Spec}(k) \end{tikzcd}$$

where f is dominant. Let $K' = k'(S')$ and for $\ell \in \mathbb{L}'$ let $\rho'_\ell = \rho_\ell|_{\operatorname{Gal}(K')}$. If $(\rho_\ell)_{\ell\in\mathbb{L}'}$ satisfies condition $\mathcal{R}(S/k)$ (resp. condition $\mathcal{S}(S/k)$), then the family $(\rho'_\ell)_{\ell\in\mathbb{L}'}$ satisfies condition $\mathcal{R}(S'/k')$ (resp. condition $\mathcal{S}(S'/k')$). Moreover, if ρ_ℓ factors via $\pi_1(S)$ and is ℓ-tame, then ρ'_ℓ factors via $\pi_1(S')$ and is ℓ-tame.

Proof. Assume that the family $(\rho_\ell)_{\ell \in \mathbb{L}'}$ satisfies condition $\mathcal{R}(S/k)$. Then there exists a dense open subscheme $U \subset S$ such that each ρ_ℓ factors through $\pi_1(U)$. Let $U' = U \times_S S'$. From the commutative diagram

$$\begin{array}{ccc} \mathrm{Gal}(K') & \longrightarrow & \mathrm{Gal}(K) \\ \downarrow & & \downarrow \\ \pi_1(U') & \longrightarrow & \pi_1(U) \end{array}$$

we see that each ρ'_ℓ factors through $\pi_1(U')$, i.e., that $(\rho'_\ell)_{\ell \in \mathbb{L}'}$ satisfies condition $\mathcal{R}(S'/k')$. Assume from now on that $(\rho_\ell)_{\ell \in \mathbb{L}'}$ satisfies condition $\mathcal{S}(S/k)$. By Lemma 5.2 (b) we can find a witness $h: V \to U$ of condition $\mathcal{S}(S/k)$ such that h is finite étale – we may need to shrink the above U. Let V' be a connected component of $V \times_U U'$. Then V' is a connected finite étale cover of U'. Let $g: V' \to V$ be the canonical map. It is enough to prove that the composition $\rho_\ell \circ h_* \circ g_*: \pi_1(V') \to G_\ell$ is ℓ-tame for every $\ell \in \mathbb{L}'$. But this is immediate from Lemma 4.5 (a), as is the last assertion. □

The following lemma describes a situation in which a family $(\rho_\ell: \mathrm{Gal}(K) \to G_\ell)_{\ell \in \mathbb{L}'}$ becomes everywhere unramified after a finite base change. In its application, all G_ℓ will be finite.

Lemma 5.6. *Assume that $(\rho_\ell)_{\ell \in \mathbb{L}'}$ satisfies condition $\mathcal{S}(S/k)$. Then there exists a finite extension k' over k, a smooth projective k' variety $\overline{W}$, and a witness $g: W \to U$ of condition $\mathcal{S}(S/k)$ such that W is a dense open subscheme of $\overline{W}$. In particular, if G_ℓ is of order prime to ℓ, then $\rho_\ell|_{\mathrm{Gal}(k(W))}$ factors through $\pi_1(\overline{W})$.*

Proof. Let $f: V \to U$ be a witness of condition $\mathcal{S}(S/k)$ for the family $(\rho_\ell)_{\ell \in \mathbb{L}'}$. By de Jong's theorem [6] there exists a finite extension k'/k, a smooth projective k'-variety $\overline{W}$, a dense open subscheme W of $\overline{W}$ and an alteration $h: W \to V$. Then the first assertion follows from Lemma 5.2 (a) for $g := f \circ h$. Fix now $\ell \in \mathbb{L}'$ and assume that ℓ does not divide the order of G_ℓ. If v denotes the discrete valuation of any codimension 1 point of $\overline{W}$, then the group $\rho_\ell \circ g_*(I_v)$ is trivial because it is pro-ℓ and of order prime to ℓ at the same time. By the purity of the branch locus it follows that $\rho_\ell \circ g_*$ factors through $\pi_1(\overline{W})$ as desired. □

Combining ramification properties with finiteness properties of fundamental groups, we obtain the following criterion for a family $(\rho_\ell: \mathrm{Gal}(K) \to G_\ell)_{\ell \in \mathbb{L}'}$ to become trivial over $\mathrm{Gal}(\widetilde{k}K')$ for some finite K'/K, provided certain finiteness conditions on $\rho_\ell(\mathrm{Gal}(K))$ hold.

Proposition 5.7. *Assume that the family $(\rho_\ell: \mathrm{Gal}(K) \to G_\ell)_{\ell \in \mathbb{L}'}$ satisfies condition $\mathcal{R}(S/k)$. If $p > 0$, then assume $(\rho_\ell: \mathrm{Gal}(K) \to G_\ell)_{\ell \in \mathbb{L}'}$ satisfies $\mathcal{S}(S/k)$. Under either of the following two conditions there exists a finite Galois extension K' of K such that for all $\ell \in \mathbb{L}'$ we have $\rho_\ell(\mathrm{Gal}(\tilde{k}K')) = \{1\}$.*

(a) *The field k is finite or k is a number field, and there exists a constant $d \in \mathbb{N}$ such that for each $\ell \in \mathbb{L}'$ the group $\rho_\ell(\mathrm{Gal}(K))$ lies in $\mathrm{Jor}_\ell(d)$.*

(b) *The field k is algebraically closed and there exists a constant $c \in \mathbb{N}$ such that for each $\ell \in \mathbb{L}'$ the group $\rho_\ell(\mathrm{Gal}(K))$ is of order at most c.*

Proof. Because of $\mathcal{R}(S/k)$ there exists a dense open subscheme U of S such that each ρ_ℓ factors through $\pi_1(U)$. Let K_ℓ be the fixed field of $\ker(\rho_\ell)$ and let $E = \prod_{\ell \in \mathbb{L}'} K_\ell$. Then K_ℓ/K is unramified along U. We have to prove that $\tilde{k}E/\tilde{k}K$ is finite.

Assume $p = 0$. In case (a) Proposition 4.1 yields that the extension $\tilde{k}E/\tilde{k}K$ is finite. In case (b) we have $\tilde{k} = k$ and thus the (geometric) fundamental group $\pi_1(U)$ is finitely generated (cf. [16, Théorème X.2.9]). Hence, independently of ℓ, there are only finitely many possibilities for the fields K_ℓ, and so E/K is finite in case (b), as well.

Assume from now on that $p > 0$. Note that in both cases (a) and (b) the order of the finite group G_ℓ is prime to ℓ for all but finitely many $\ell \in \mathbb{L}'$. By Lemma 5.6 there exists a finite extension k'/k and a finite extension F/K and a smooth projective k'-variety $\overline{W}$ with function field F such that the extension $K_\ell F/F$ is unramified along $\overline{W}$ for almost all $\ell \in \mathbb{L}'$. In case (a) Proposition 4.1 yields that $\tilde{k}EF/\tilde{k}F$ is finite. Hence $\tilde{k}E/\tilde{k}K$ must be finite. Finally, in case (b) the group $\pi_1(\overline{W})$ is finitely generated (cf. [17, II.2.3.1]), and thus E/K must be finite. $\square$

The following independence criterion is the main result of this section:

Theorem 5.8. *Assume that k is algebraically closed. Assume that the following conditions* (a) *and* (b) *are satisfied.*

(a) *The family $(\rho_\ell)_{\ell \in \mathbb{L}'}$ satisfies $\mathcal{R}(S/k)$, and it satisfies $\mathcal{S}(S/k)$ if $p > 0$.*

(b) *There exists a constant $c \in \mathbb{N}$ and a finite Galois extension K'/K such that for all $\ell \in \mathbb{L}'$ one has $\rho_\ell(\mathrm{Gal}(K')) \in \Sigma_\ell(c)$.*

Then there exists a finite Galois extension E/K containing K' such that $\mathrm{Gal}(E/K')$ *is abelian and such that the following holds true.*

(i) *For every $\ell \in \mathbb{L}'$ the group $\rho_\ell(\mathrm{Gal}(E))$ lies in $\Sigma_\ell(c)$ and is generated by its ℓ-Sylow subgroups; if $\ell > c$, then the group $\rho_\ell(\mathrm{Gal}(E))$ is generated by the ℓ-Sylow subgroups of $\rho_\ell(\mathrm{Gal}(K))$.*

(ii) *The restricted family $(\rho_\ell|_{\mathrm{Gal}(E)})_{\ell \in \mathbb{L}' \smallsetminus \{2,3\}}$ is independent and $(\rho_\ell)_{\ell \in \mathbb{L}'}$ is almost independent.*

Proof. Let $G_\ell = \rho_\ell(\mathrm{Gal}(K'))$ for all $\ell \in \mathbb{L}'$. The group $\overline{G_\ell} := G_\ell/G_\ell^+$ is finite and of order prime to ℓ. Denote by $\pi_\ell \colon G_\ell \to \overline{G_\ell}$ the natural projection. Let K'_ℓ be the fixed field in K_s of the kernel of the composite morphism

$$\mathrm{Gal}(K') \xrightarrow{\rho_\ell} G_\ell \xrightarrow{\pi_\ell} \overline{G_\ell}.$$

As G_ℓ lies in $\Sigma_\ell(c)$, so does its quotient $\overline{G}_\ell$ by Lemma 3.3 (b). Now any group in $\Sigma_\ell(c)$ of order prime to ℓ is abelian of order at most c, and thus the latter holds for $\overline{G}_\ell$. Thus K'_ℓ/K' is an abelian Galois extension of degree prime to ℓ and $\le c$. Moreover, as G_ℓ^+ is a characteristic subgroup of G_ℓ, it follows that the finite extension K'_ℓ/K is Galois. Thus the compositum $E = \prod_{\ell \in \mathbb{L}'} K'_\ell$ is Galois over K, and $\mathrm{Gal}(E/K')$ is an abelian group annihilated by $c!$. Let S' denote the normalization of S in K' and S'' a dense regular open subscheme of S'. Then $\rho_\ell|_{\mathrm{Gal}(K')}$ satisfies condition $\mathcal{R}(S''/k)$ and it satisfies condition $\mathcal{S}(S''/K)$ if $p > 0$ (cf. Proposition 5.5). From Proposition 5.7 (b) we conclude that E/K is finite.

We turn to the proof of (i). For every $\ell \in \mathbb{L}'$, the group $\rho_\ell(\mathrm{Gal}(E))$ is normal in G_ℓ, and hence it lies in $\Sigma_\ell(c)$ by Lemma 3.3. Let $M_\ell = \rho_\ell(\mathrm{Gal}(E))$. By construction, $M_\ell \triangleleft G_\ell^+$, and G_ℓ/M_ℓ is abelian and killed by $c!$ because it is a quotient of $\mathrm{Gal}(E/K')$. Thus G_ℓ^+/M_ℓ is an abelian ℓ-group which is killed by $c!$; if $\ell > c$ then this implies $G_\ell^+ = M_\ell$. To establish (i) it now suffices to prove that $M_\ell = M_\ell^+$ for all $\ell \in \mathbb{L}'$ with $\ell \le c$. Clearly, M_ℓ/M_ℓ^+ is abelian, and hence G_ℓ^+/M_ℓ^+ is a finite solvable group that is generated by its ℓ-Sylow subgroups. In addition, the group G_ℓ^+/M_ℓ^+ lies in $\Sigma_\ell(c)$, and therefore it must be an ℓ-group. Thus M_ℓ/M_ℓ^+ is an ℓ-group as well, and by the definition of M_ℓ^+, we deduce $M_\ell = M_\ell^+$. Hence part (i) holds true.

We now prove (ii). Denote by Ξ_ℓ the class of those finite groups which are either a finite simple group of Lie type in characteristic ℓ or isomorphic to $\mathbb{Z}/\ell$. The conditions in (i) imply that every simple quotient of $\rho_\ell(\mathrm{Gal}(E))$ lies in Ξ_ℓ. But now for any $\ell, \ell' \ge 5$ such that $\ell \ne \ell'$ one has $\Xi_\ell \cap \Xi_{\ell'} = \varnothing$ (cf. [35, Theorem 5], [1], [24]). The first part of (ii) now follows from [35, Lemme 2]. The second part follows from the first, the definition of almost independence and from [35, Lemme 3]. □

Remark 5.9. We would like to point out that hypothesis (a) in the proof of Theorem 5.8 can be weakened considerably. For this we denote for a continuous homomorphism $\rho_\ell\colon \mathrm{Gal}(K) \to G_\ell$ by Q_ℓ the maximal normal pro-ℓ subgroup of $\rho_\ell(\mathrm{Gal}(K))$, and by $\widetilde{\rho}_\ell$ the composite homomorphism

$$\mathrm{Gal}(K) \xrightarrow{\rho_\ell} \rho_\ell(\mathrm{Gal}(K)) \longrightarrow \rho_\ell(\mathrm{Gal}(K))/Q_\ell.$$

If ρ_ℓ is an ℓ-adic representation, then $\widetilde{\rho}_\ell$ is simply the semisimplification of the mod ℓ reduction of ρ_ℓ. The proof of Theorem 5.8 only needs that the family $(\widetilde{\rho}_\ell)_{\ell\in\mathbb{L}'}$ satisfies condition $\mathcal{R}(S/k)$ or condition $\mathcal{S}(S/k)$, if $p > 0$, respectively, because this weaker hypothesis suffices for the finiteness of E/K.

We chose to work with conditions $\mathcal{R}(S/k)$ and $\mathcal{S}(S/k)$ as introduced in Definition 5.1, since they seem most natural for the motivic families we consider in Theorem 1.2. These conditions are established in Proposition 5.4 and Corollary 7.4. For other purposes, the variant of Definition 5.1 using $(\widetilde{\rho}_\ell)_{\ell\in\mathbb{L}'}$ instead might be useful: There are infinitely ramified ℓ-adic representations of curves over finite fields, that can be constructed following [31]. Families of such will never satisfy $\mathcal{R}(S/k)$. Also, if a family $(\tau_\ell)_{\ell\in\mathbb{L}'}$ satisfies $\mathcal{R}(S/k)$ and if ρ_ℓ is an extension of τ_ℓ by itself where the extension class is ramified at a divisor depending on ℓ, then $\mathcal{R}(S/k)$ might fail for $(\rho_\ell)_{\ell\in\mathbb{L}'}$.

6. Effective semistability of families $(\rho_{\ell,X}^{(q)})_{\ell\in\mathbb{L}'}$ for $p > 0$

Let k be a perfect field of characteristic $p > 0$, S a separated algebraic scheme over k with function field $K = k(S)$, and let X/K be a smooth projective variety. Let $q \in \mathbb{N}$. The main result of this section, Corollary 6.3, gives an effective proof of condition $\mathcal{S}(S/k)$ for the family $(\rho_{\ell,X}^{(q)})_{\ell\in\mathbb{L}'}$. We shall describe explicit finite Galois extensions K' of K, such that for all $\ell \in \mathbb{L}'$ the representation $\rho_{\ell,X}^{(q)}|_{\mathrm{Gal}(K')}$ is ℓ-tame, cf. also Remark 6.4. Our proof uses a reduction to $k = \mathbb{F}_p$ in which case we can apply the Weil conjectures and an ℓ-independence result on tameness from [7], both due to Deligne. Our method sheds no light on the existence of a semistable geometric model of X/K over some smooth proper scheme S/k' with function field K'. Such an approach is given in [30]. However, in [30] it might be hard to find an effective description of K'.

Throughout this section we let $p > 0$ be a prime number, and we set $\mathbb{L}' = \mathbb{L} \smallsetminus \{p\}$. We use the subscript 0 for fields and separated algebraic schemes that are finitely generated over $\mathbb{F}_p$. In particular, $S_0/\mathbb{F}_p$ will be a smooth variety with function field $K_0 = \mathbb{F}_p(S_0)$.

For every open subscheme U_0 of S_0 and every closed point $u \in U_0$, let $k(u)$ be the (finite) residue field of u, and let $D(u) \subset \pi_1(U_0)$ be the corresponding decomposition group (defined only up to conjugation). Denote by $\mathrm{Fr}_u \in D(u)$ the preimage under the canonical isomorphism $D(u) \xrightarrow{\sim} \mathrm{Gal}(k(u))$ of the arithmetic Frobenius

$$\sigma_u \colon \widetilde{k(u)} \to \widetilde{k(u)}, \qquad x \mapsto x^{\frac{1}{|k(u)|}}.$$

Note that within $\pi_1(U_0)$, the automorphism Fr_u is also defined only up to conjugation. The following proposition is an immediate consequence of the Weil conjectures proved by Deligne.

Proposition 6.1. *Let X_0/K_0 be a smooth projective variety. There exists a dense open subscheme U_0 of S_0 such that for every $q \in \mathbb{N}$ and every $\ell \in \mathbb{L}'$ the representation $\rho^{(q)}_{\ell,X_0}$ factors through $\pi_1(U_0)$ and such that the family of representations $(\rho^{(q)}_{\ell,X_0})_{\ell\in\mathbb{L}'}$ is strictly compatible and pure of weight q, that is: For every closed point $u \in U_0$ the characteristic polynomial $p_u(T)$ of $\rho^{(q)}_{\ell,X_0}(\mathrm{Fr}_u)$ has integral coefficients, is independent of $\ell \in \mathbb{L}'$, and the roots of $p_u(T)$ all have absolute value $|k(u)|^{q/2}$.*

Proof. There exists a dense open subscheme U_0 of S_0 and a projective U_0-scheme $f \colon \mathcal{X}_0 \to U_0$ such that $\mathcal{X}_0 \times_{U_0} \mathrm{Spec}(K_0) = X_0$ (cf. [15, 8.8.2] and [15, 8.10.5 (v) and (xiii)]). By the theorem of generic smoothness, after shrinking U_0 and $\mathcal{X}_0$, we can assume that f is smooth.

Let $q \geq 0$, $u \in U_0$. Define $k := k(u)$ and $X_u := \mathcal{X}_0 \times_{U_0} \mathrm{Spec}(k)$. Then for every $\ell \in \mathbb{L}'$ the étale sheaf $R^q f_* \mathbb{Z}_\ell$ is lisse and compatible with any base change (cf. [28, VI.2, VI.4]). Thus $\rho^{(q)}_{\ell,X_0}$ factors through $\pi_1(U_0)$, and furthermore it follows that $H^q(X_{0,\widetilde{K}}, \mathbb{Q}_\ell)$ can be identified with $H^q(X_{u,\tilde{k}}, \mathbb{Q}_\ell)$ in a way compatible with the Galois actions. The assertion now follows by Deligne's theorem on the Weil conjectures (cf. [8, Theorem 1.6]). □

Lemma 6.2. *Let X_0/K_0 be a smooth projective variety, and let q be in $\mathbb{Z}$. Suppose that for some prime $\ell_0 \geq 3$ in $\mathbb{L}'$ there is a $\mathbb{Z}_{\ell_0}$-lattice Λ of the $\mathbb{Q}_{\ell_0}$-representation space underlying $\rho^{(q)}_{\ell_0,X_0}$ that is stabilized by* $\mathrm{Gal}(K_0)$ *and such that* $\mathrm{Gal}(K_0)$ *acts trivially on $\Lambda/\ell_0\Lambda$. Then for all $\ell \in \mathbb{L}'$, the representation $\rho^{(q)}_{\ell,X_0}$ is ℓ-tame.*

Proof. Let C be any smooth curve over $\mathbb{F}_p$ and let $\varphi \colon C \to U_0$ be any non-constant morphism. Denote by $P(C)$ a smooth projective model of C, and by $\varphi_* \colon \pi_1(C) \to \pi_1(U_0)$ the homomorphism on fundamental groups induced by φ. We claim that the representation

$$(\rho^{(q)}_{\ell,X_0} \circ \varphi_*)^{\mathrm{ss}} \tag{2}$$

is semistable[4)] at all places of $P(C) \smallsetminus C$ and for all $\ell \in \mathbb{L}'$. Having shown this for all pairs (C, φ), the assertion of the lemma is deduced as follows: Passing from an ℓ-adic representation

4) We call a representation of $\pi_1(C)$ *semistable* at v if the Frobenius semisimplified Weil–Deligne representation at v is unramified when restricted to the Weil group.

to its semisimplification does not affect ℓ-tameness, and so $\rho^{(q)}_{\ell,X_0} \circ \varphi_*$ will be ℓ-tame for all ℓ. The present lemma now is immediate from this and Proposition 4.6, which is a variant of a result of Kerz–Schmidt–Wiesend.

It remains to prove the claim. By Proposition 6.1, the representations (2) are pure of weight q and semisimple for all $\ell \in \mathbb{L}'$, and they form a strictly compatible family of representation of $\pi_1(C)$. By [7, Théorème 9.8], it follows that in fact for any $v \in P(C)$ the representation $r_{v,\ell}$ underlying the Weil–Deligne representation $(r_{v,\ell}, N_{v,\ell})$ of the restriction of (2) to a decomposition group at v is independent of ℓ. Thus it will suffice to show that r_{v,ℓ_0} is unramified for all $v \in P(C)$.

We consider the representations (2) for $\ell = \ell_0$ as an action of $\pi_1(C)$ on the lattice Λ that is trivial modulo $\ell_0\Lambda$. Any filtration of $\Lambda \otimes_{\mathbb{Z}_{\ell_0}} \mathbb{Q}_{\ell_0}$ that is preserved by the action of $\pi_1(C)$ induces a filtration of Λ. Denote by Λ_C the induced lattice for (2). Then it follows that the induced action of $\pi_1(C)$ on $\Lambda_C/\ell_0\Lambda_C$ is trivial. Let $n = \operatorname{rank}\Lambda$. In the following we frequently identify $\operatorname{Aut}_{\mathbb{Z}_{\ell_0}}(\Lambda)$ with $\mathrm{GL}_n(\mathbb{Z}_{\ell_0})$. We define

$$\mathrm{GL}^1_n(\mathbb{Z}_{\ell_0}) := \ker(\operatorname{Aut}(\Lambda_C) \to \operatorname{Aut}(\Lambda_C/\ell_0\Lambda_C)).$$

Since $\ell_0 > 2$, we have pairwise inverse ℓ_0-adic exponential and logarithm maps

$$\exp_{\ell_0}\colon \ell_0 M_n(\mathbb{Z}_{\ell_0}) \to \mathrm{GL}^1_n(\mathbb{Z}_{\ell_0}) \quad \text{and} \quad \log_{\ell_0}\colon \mathrm{GL}^1_n(\mathbb{Z}_{\ell_0}) \to \ell_0 M_n(\mathbb{Z}_{\ell_0}),$$

given by the usual power series expressions

$$X \mapsto \sum_{m\geq 0} \frac{1}{m!} X^m \quad \text{and} \quad (1+A) \mapsto \sum_{m \geq 1} \frac{(-1)^{m+1}}{m} A^m$$

(cf. [4, III.7.2], [4, III.7.6]). It follows that $\log_{\ell_0}$ converges on the image of (2). This image being in $\mathrm{GL}^1_n(\mathbb{Z}_{\ell_0})$, following the standard argument (cf. [36, Corollary 4.2.2]) shows that r_{v,ℓ_0} is unramified: By the continuity of r_{v,ℓ_0} (with the target carrying the discrete topology), any element A in the image of inertia at v has finite order. Thus $A \in \mathrm{GL}^1_n(\mathbb{Z}_{\ell_0})$ satisfies $A^m = 1$ for some $m \in \mathbb{N}$. One deduces $0 = \log_{\ell_0} A^m = m \log_{\ell_0} A$ which gives $\log_{\ell_0} A = 0$ and thus $A = \exp_{\ell_0}(\log_{\ell_0} A) = 1_n$. □

For a smooth projective variety X/k we have the following effective ℓ-tameness result for the family $(\rho^{(q)}_{\ell,X})_{\ell\in\mathbb{L}'}$.

Corollary 6.3. *Let k be perfect of characteristic $p > 0$, let S/k be a smooth variety with function field K, and let X/K be a smooth projective variety. Choose a prime $\ell_0 \geq 3$ and a* $\mathrm{Gal}(K)$*-invariant lattice Λ of $\rho^{(q)}_{\ell_0,X}$, and denote by K' the fixed field of the kernel of $\rho^{(q)}_{\ell_0,X}$* $\pmod{\ell_0\Lambda}$. *Then there exists a regular k-scheme U' with function field K' such that for all $\ell \in \mathbb{L}'$ the family $(\rho^{(q)}_{\ell,X}|_{\mathrm{Gal}(K')})_{\ell\in\mathbb{L}'}$ factors via $\pi_1(U')$ and is ℓ-tame.*

In particular, $(\rho^{(q)}_{\ell,X})_{\ell\in\mathbb{L}'}$ satisfies condition $\mathcal{S}(S/k)$

Proof. Suppose first that $k = \mathbb{F}_p$ is finite, so that $X/K = X_0/K_0$ in the notation above. Let $U_0 \subset S$ be as in Proposition 6.1 for $(\rho^{(q)}_{\ell,X})_{\ell\in\mathbb{L}'}$, let U'_0 be its normalization in K'. By generic étaleness, we may shrink U_0 such that $U'_0 \to U_0$ is étale, and hence U'_0/k is smooth. Using Lemma 6.2, the second assertion follows for $U' = U'_0$. The first is clear from the construction.

Let now k be any perfect field of characteristic p, and let Λ and K' be as in the corollary. By standard arguments from algebraic geometry, there exist an absolutely finitely generated subfield K_0 of K and a smooth projective scheme X_0 over K_0 such that

$$X = X_0 \times_{\operatorname{Spec} K_0} \operatorname{Spec} K.$$

Then $(\rho_{\ell,X}^{(q)})_{\ell\in\mathbb{L}'} = (\rho_{\ell,X_0}^{(q)}|_{\mathrm{Gal}(K)})_{\ell\in\mathbb{L}'}$ for the canonical map $\mathrm{Gal}(K) \to \mathrm{Gal}(K_0)$. Since Λ is stabilized by $\mathrm{Gal}(K)$, we may assume by passing from K_0 to a finite extension contained in K that Λ is stabilized by $\mathrm{Gal}(K_0)$. If we furthermore define $K_0' \supset K_0$ as the fixed field of the kernel of $\rho_{\ell_0,X_0}^{(q)} \pmod{\ell_0\Lambda}$, then $K_0' \subset K'$.

Let S_0 be a smooth $\mathbb{F}_p$-variety with function field K_0. Let $U_0 \subset S_0$ and U_0' be as in the first paragraph of the proof, and assume without loss of generality that both are affine $U_0 = \operatorname{Spec} A_0$ and $U_0' = \operatorname{Spec} A_0'$. Then we have that $kA_0 \subset K$ and $kA_0' \subset K'$ are finitely generated k-algebras. Because k is perfect, they are generically smooth over k and $kA_0 \subset kA_0'$ is generically étale. We can choose $U \subset \operatorname{Spec}(kA_0)$ dense open such that $U \subset S$ and such that the normalization U' of U in K' is étale over U; note that U' is an open subscheme of $\operatorname{Spec}(kA_0')$. The corollary now follows from Proposition 5.5 applied to $U' \to U_0'$ over $\operatorname{Spec} k \to \operatorname{Spec} \mathbb{F}_p$. $\square$

Remark 6.4. We would like to point out the parallel between Corollary 6.3 and Example 5.3. In both cases we select a prime number $\ell_0 \geq 3$. In the effective part of Corollary 6.3 we enlarged K to K' so that $\mathrm{Gal}(K')$ acts trivially on $\Lambda/\ell_0\Lambda$ via the representation $\rho_{\ell_0,X}^{(q)}$ for a $\mathrm{Gal}(K)$-stable lattice Λ. Then we could use the uniformity provided by automorphic representations (after restricting ourselves to the case where k is finite by standard arguments, and after restricting the $\rho_{\ell,X}^{(q)}$ to any curve) from Lemma 6.2, to deduce that for all ℓ, all ramification of $\rho_{\ell,X}$ was ℓ-tame over K'. The semistability of an automorphic representation, and hence the ℓ-tameness of the associated compatible system of ℓ-adic Galois representations, spreads from a single ℓ_0 to all ℓ.

In Example 5.3 we set $K' = K(A[\ell_0])$, which again is the minimal choice so that $\mathrm{Gal}(K')$ acts trivially on the quotient $T_{\ell_0}(A)/\ell_0 T_{\ell_0}(A)$ for the lattice $T_{\ell_0}(A)$ from the Tate-module. Then we use the semistability of the Néron model $\mathcal{N}$ of the abelian variety A over K' over any discrete valuation ring, that is implied by a condition at a single prime ℓ_0. We deduce the ℓ-tameness of $T_\ell(A)$ over K', uniformly for each prime ℓ. In both cases, the field K' is defined in the same way. For general X as in Corollary 6.3, it seems unlikely that X always acquires some geometric semistability over K', as it is the case for X an abelian variety.

7. Reduction steps and the proof of Theorem 1.2

Throughout this section let k be a field of characteristic $p \geq 0$, let $\mathbb{L}' = \mathbb{L} \smallsetminus \{p\}$ and let S/k be a smooth variety with function field K. By X/K we denote a separated algebraic scheme, and by $\rho_{\ell,X}$ the representation of $\mathrm{Gal}(K)$ *on* $\bigoplus_{q\geq 0}(H_c^q(X_{\widetilde{K}}, \mathbb{Q}_\ell) \oplus H^q(X_{\widetilde{K}}, \mathbb{Q}_\ell))$.

In this section, we give the necessary reduction steps to deduce hypotheses (a) and (b) of Theorem 5.8 for the family $(\rho_{\ell,X})_{\ell\in\mathbb{L}'}$. Thereby we shall complete the proof of Theorem 1.2. Recall that hypothesis (a) is condition $\mathcal{S}(S/k)$, and that hypothesis (b) is the uniform containment $\rho_{\ell,X}(\mathrm{Gal}(K')) \in \Sigma_\ell(c)$ for all ℓ for some finite extension $K' \supset K$.

As a preparation to establish $\mathcal{S}(S/k)$ we introduce the following notion.

Definition 7.1. For a given representation $\rho_\ell\colon \mathrm{Gal}(K) \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ we define its *strict semisimplification* ρ_ℓ^{sss} as the direct sum over the irreducible subquotients of ρ_ℓ where each isomorphism type occurs with multiplicity one.

Note that $\rho_\ell^{\mathrm{sss}}(\mathrm{Gal}(K)) = \rho_\ell^{\mathrm{ss}}(\mathrm{Gal}(K))$ where ρ_ℓ^{ss} denotes the usual semisimplification of ρ_ℓ.

Lemma 7.2. *For every $\ell \in \mathbb{L}'$ let ρ_ℓ and ρ'_ℓ be representations* $\mathrm{Gal}(K) \to \mathrm{GL}_n(\mathbb{Q}_\ell)$. *Suppose that the families $(\rho_\ell)_{\ell\in\mathbb{L}'}$ and $(\rho'_\ell)_{\ell\in\mathbb{L}'}$ both satisfy condition $\mathcal{R}(S/k)$. Suppose that one of the following two assertions is true:*

(a) $\rho_\ell^{\mathrm{sss}} = (\rho'_\ell)^{\mathrm{sss}}$ *for all* $\ell \in \mathbb{L}'$,

(b) ρ_ℓ *is a direct summand of* ρ'_ℓ *for all* $\ell \in \mathbb{L}'$.

Then the following hold: If the family $(\rho'_\ell)_{\ell\in\mathbb{L}'}$ satisfies $\mathcal{S}(S/k)$, then so does $(\rho_\ell)_{\ell\in\mathbb{L}'}$.

Proof. The proof under hypothesis (a) is an immediate consequence of the simple fact that the kernel of $\rho_\ell(\mathrm{Gal}(K)) \to \rho_\ell^{\mathrm{ss}}(\mathrm{Gal}(K)) = \rho_\ell^{\mathrm{sss}}(\mathrm{Gal}(K))$ is a pro-ℓ-group.

Under hypothesis (b) the proof is trivial. □

The following important result is taken from the Seminaire Bourbaki talk of Berthelot on de Jong's alteration technique (cf. [3, Theorem 6.3.2])

Theorem 7.3. *Let X be a separated algebraic scheme over K. Then there exists a finite extension k'/k, a finite separable extension K'/Kk' and a finite set of smooth projective varieties $\{Y_i\}_{i=1,\dots,r}$ over K' such that for all $\ell \in \mathbb{L}'$ the representation $(\rho_{\ell,X}|_{\mathrm{Gal}(K')})^{\mathrm{sss}}$ is a direct summand of $(\bigoplus_i \rho_{\ell,Y_i})^{\mathrm{sss}}$.*

The following result is an immediate consequence of Theorem 7.3, Lemma 7.2 and Corollary 6.3.

Corollary 7.4. *Let k be perfect of characteristic $p > 0$; recall that $K = k(S)$. Then for every separated algebraic K-scheme X the family $(\rho_{\ell,X})_{\ell\in\mathbb{L}'}$ satisfies condition $\mathcal{S}(S/k)$.*[5)]

Proof of Theorem 7.3. For completeness we provide details of the proof in [3]. For $? \in \{c, \varnothing\}$ we denote by $\rho_{\ell,X,?}$ the representation of $\mathrm{Gal}(K)$ on $\bigoplus_{q\geq 0}(H^q_?(X_{\widetilde{K}}, \mathbb{Q}_\ell))$. It suffices to prove the theorem separately for the families $(\rho_{\ell,X,?})_{\ell\in\mathbb{L}'}$. We also note that whenever it is convenient, we are allowed (by passing from K to a finite extension) to assume that X is geometrically reduced over K. This is so because $H^q_?(X_{\widetilde{K}}, \mathbb{Q}_\ell) \cong H^q_?(X_{\widetilde{K},\mathrm{red}}, \mathbb{Q}_\ell)$ for any $q \in \mathbb{Z}$ and $? \in \{\varnothing, c\}$. We first consider the case of cohomology with compact supports. The proof proceeds by induction on $\dim X$.

5) Not assuming that k is perfect, one can show that after a finite field extension k' of k and a corresponding base change S' of S, condition $\mathcal{S}(S'/k')$ holds.

The scheme $X_{\widetilde{K}}$ is generically smooth. After passing from K to a finite extension, we can find a dense open subscheme $U \subset X$ that is smooth over K. By the long exact cohomology sequence with supports (cf. [28, Remark III.1.30]) we have for any ℓ an exact sequence

$$\cdots \to H^i_c(U_{\widetilde{K}}, \mathbb{Q}_\ell) \to H^i_c(X_{\widetilde{K}}, \mathbb{Q}_\ell) \to H^i_c((X \smallsetminus U)_{\widetilde{K}}, \mathbb{Q}_\ell) \to \cdots,$$

so that for all ℓ the representation $(\rho_{\ell,X,c})^{\mathrm{sss}}$ is a direct summand of $(\rho_{\ell,U,c})^{\mathrm{sss}} \oplus (\rho_{\ell,X\smallsetminus U,c})^{\mathrm{sss}}$. By induction hypothesis, it thus suffices to treat the case that U is smooth over K. By induction hypothesis, it is also sufficient to replace U by any smaller dense open subscheme, and it is clearly also sufficient to treat the case where U is in addition geometrically irreducible.

By de Jong's theorem on alterations (cf. [6, Theorems 4.1, 4.2]), after passing from K to a finite extension, we can find a smooth projective scheme Y, an open subscheme U' of Y and an alteration $\pi\colon U' \to U$. By replacing K yet another time by a finite extension, we can assume that $U' \to U$ is generically finite étale. And now we pass to an open subscheme V of U and to $V' := \pi^{-1}(V) \subset U'$ such that $V' \to V$ is finite étale. By the induction hypothesis applied to $Y \smallsetminus V'$ and again the long exact cohomology sequence for cohomology with supports, we find that the assertion of the theorem holds true for the family $(\rho_{\ell,V',c})^{\mathrm{sss}}_{\ell\in\mathbb{L}'}$. From now on π denotes the restriction to V' and $\mathbb{Q}_{\ell,X}$ will be the constant sheaf $\mathbb{Q}_\ell$ on any scheme X. Since π is finite étale, say of degree d, there exists a trace morphism $\mathrm{Trace}_\pi\colon \pi_*\mathbb{Q}_{\ell,V'} \to \mathbb{Q}_{\ell,V}$ whose composition with the canonical morphism $\mathbb{Q}_{\ell,V} \to \pi_*\mathbb{Q}_{\ell,V'}$ is multiplication by d (cf. [28, Lemma V.1.12]). In particular, we have that the constant sheaf $\mathbb{Q}_{\ell,V}$ is a direct summand of $\pi_*\mathbb{Q}_{\ell,V'}$. Since $H^i_c(V'_{\widetilde{K}}, \mathbb{Q}_\ell) \cong H^i_c(V_{\widetilde{K}}, \pi_*\mathbb{Q}_\ell)$, we deduce that $(\rho_{\ell,V,c})^{\mathrm{sss}}$ is a direct summand of $(\rho_{\ell,V',c})^{\mathrm{sss}}$, and this completes the induction step.

Now we turn to the case $? = \varnothing$. The case when X is smooth over K but not necessarily projective is reduced, by Poincaré duality, to the case of compact supports. If X is connected, one has $H^q(X_{\widetilde{K}}, \mathbb{Q}_\ell) \cong H^{2d-q}_c(X_{\widetilde{K}}, \mathbb{Q}_\ell(d))^\vee$ for $d = \dim X$ (cf. [28, Corollary VI.11.12]), and one can reduce to the connected case by considering the connected components of X separately.

Suppose now that X is an arbitrary separated algebraic scheme over K. By what we said above, we may assume that X is geometrically reduced. Again we perform an induction on $\dim X$. The first step is a reduction to the case where X is irreducible, which may be thought of as an induction by itself. Suppose $X = X_1 \cup X_2$ where X_1 is an irreducible component of X and X_2 is the closure of $X \smallsetminus X_1$. Consider the canonical morphism $f\colon X_1 \sqcup X_2 \to X$. It yields a short exact sequence of sheaves

$$0 \to \mathbb{Q}_{\ell,X} \to f_*\mathbb{Q}_{\ell,X_1\sqcup X_2} \to \mathcal{F} \to 1 \tag{3}$$

where $\mathcal{F}$ is a sheaf on X. Consider the inclusion $i\colon X_0 \hookrightarrow X$ for $X_0 := X_1 \cap X_2$. We claim that $\mathcal{F} \cong i_*\mathbb{Q}_{\ell,X_0}$. To see this observe first that if we compute the pullback of the sequence along the open immersion $j\colon X \smallsetminus X_0 \hookrightarrow X$, then $\mathcal{F}$ vanishes and the morphism on the left becomes an isomorphism. In particular, $\mathcal{F}$ is supported on X_0. To compute the pullback along the closed immersion i, we may apply proper base change, since f is proper. But now the restriction of f to X_0 is simply the trivial double cover $X_0 \sqcup X_0 \twoheadrightarrow X_0$, so that $i^*\mathcal{F} \cong \mathbb{Q}_{\ell,X_0}$. This proves the claim because $\mathcal{F} \cong i_*i^*\mathcal{F}$, as $\mathcal{F}$ is supported on X_0. By an inductive application of the long exact cohomology sequences to sequences like (3), it suffices to prove the theorem for schemes X that are geometrically integral and separated algebraic over K. In this case, the proof follows by resolving X by a smooth hypercovering $X_\bullet$ see [6, p. 51], [9, 6.2.5] and the proof of [3, Theorem 6.3.2]. Since the hypercovering yields a spectral sequence that computes the cohomology of X in terms of the cohomologies of the smooth X_i, for all ℓ, and since only those X_i with $i \leq 2\dim X$, contribute to X, the induction step is complete because we have reduced the case of arbitrary X to lower dimensions and to smooth X_i. □

We now consider hypothesis (b) of Theorem 5.8. One has the following simple reduction result.

Lemma 7.5. *Let* $(\rho_\ell\colon \mathrm{Gal}(K_1) \to G_\ell)_{\ell\in\mathbb{L}'}$ *be a family of Galois representations, and consider the diagram of fields*

$$\begin{tikzcd} K_1 \arrow[r, hook] & K_2 \\ k_1 \arrow[u, hook] \arrow[r, hook] & k_2 \arrow[u, hook] \end{tikzcd}$$

inside an algebraic closure $\widetilde{K}_2$ *of* K_2*, where* k_i *is algebraically closed and* K_i *is finitely generated over* k_i *for* $i = 1, 2$*. Then* $(*_1)$ *implies* $(*_2)$ *for the following assertion depending on* i*:*

$(*_i)$ *There exists a constant* $c \in \mathbb{N}$ *and a finite Galois extension* K_i'/K_i *such that for all* $\ell \in \mathbb{L}'$ *one has* $\rho_\ell(\mathrm{Gal}(K_i')) \in \Sigma_\ell(c)$*.*

Proof. Consider the following homomorphisms of Galois groups, given by restriction:

$$\mathrm{Gal}(K_2K_1') \xrightarrow{r_3} \mathrm{Gal}(k_2K_1') \xrightarrow{r_2} \mathrm{Gal}((k_2 \cap \widetilde{K}_1)K_1') \xrightarrow{r_1} \mathrm{Gal}(K_1').$$

One easily verifies that r_1 has closed normal image, r_2 is surjective, and r_3 has open image. Let E/k_2K_1' be a finite Galois extension such that $r_3(\mathrm{Gal}(K_2K_1'))$ contains $\mathrm{Gal}(E)$, and define K_2'' as the fixed field in $\widetilde{K}_2$ of $r_3^{-1}(\mathrm{Gal}(E))$, i.e., as EK_2. Applying Corollary 3.4 to r_1, r_2 and r_3 shows that the conclusion of $(*_2)$ holds with K_2'' in place of K_2'. The lemma follows by another application of Corollary 3.4 for K_2' the Galois closure of K_2'' over K_2. □

Corollary 7.6. *For* k *be algebraically closed, hypothesis* (b) *of Theorem* 5.8 *holds for the family* $(\rho_{\ell,X})_{\ell\in\mathbb{L}'}$*.*

Proof. Let k_0 be the prime field of K, let $K_0 \subset K$ be an absolutely finitely generated subfield field, and let X_0/K_0 be a separated algebraic scheme with $X = X_0 \times_{\mathrm{Spec}\, K_0} \mathrm{Spec}\, K$. Then we have $(\rho_{\ell,X})_{\ell\in\mathbb{L}'} = (\rho_{\ell,X_0}|_{\mathrm{Gal}(K)})_{\ell\in\mathbb{L}'}$. By [22, Section 3] there exists $n \in \mathbb{N}$ such that $\dim(\rho_{\ell,X_0}) \leq n$ for all $\ell \in \mathbb{L}'$. For each ℓ we apply Theorem 3.6 to ρ_{ℓ,X_0}, to obtain a short exact sequence

$$1 \to M_\ell \to \rho_{\ell,X_0}(\mathrm{Gal}(K_0)) \to H_\ell \to 1$$

with $H_\ell \in \mathrm{Jor}_\ell(J'(n))$ and $M_\ell \in \Sigma_\ell(2^{n-1})$. Because of Corollary 7.4 and Proposition 5.4, there exists a smooth variety S_0 over k_0 with function field $K_0 = k_0(S_0)$ such that $(\rho_{\ell,X_0})_{\ell\in\mathbb{L}'}$ satisfies $\mathcal{S}(S_0/k_0)$ if $p > 0$, or satisfies $\mathcal{R}(S_0/k_0)$ if $p = 0$. Then Proposition 5.7 (a) yields a finite Galois extension K_0'/K_0 such that $\rho_{\ell,X_0}(\mathrm{Gal}(\tilde{k}_0K_0'))$ is a closed normal subgroup of M_ℓ for all $\ell \in \mathbb{L}'$. Corollary 3.4 together with Lemma 7.5 now provide a finite Galois extension K' of K such that for all $\ell \in \mathbb{L}'$ one has $\rho_\ell(\mathrm{Gal}(K')) \in \Sigma_\ell(2^{n-1})$, and so hypothesis (b) of Theorem 5.8 is satisfied. □

Using Proposition 5.4, Corollary 7.4 and Corollary 7.6, Theorem 5.8 yields our main theorem:

Theorem 7.7. *Let k be a field of characteristic $p \geq 0$ and let K/k be a finitely generated extension. Let X/K be a separated algebraic scheme. Then there exists a finite extension E/K and a constant $c \in \mathbb{N}$ with the following properties:*

(i) *For every $\ell \in \mathbb{L}'$ the group $\rho_{\ell,X}(\mathrm{Gal}(\tilde{k}E))$ lies in $\Sigma_\ell(c)$ and is generated by its ℓ-Sylow subgroups.*

(ii) *The family $(\rho_{\ell,X}|_{\mathrm{Gal}(\tilde{k}E)})_{\ell\in\mathbb{L}'\smallsetminus\{2,3\}}$ is independent and the family $(\rho_{\ell,X}|_{\mathrm{Gal}(\tilde{k}K)})_{\ell\in\mathbb{L}'}$ is almost independent.*

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Gebhard Böckle, Computational Arithmetic Geometry, IWR (Interdisciplinary Center for Scientific Computing), University of Heidelberg, Im Neuenheimer Feld 368, 69120 Heidelberg, Germany
e-mail: gebhard.boeckle@iwr.uni-heidelberg.de

Wojciech Gajda, Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61614 Poznań, Poland
e-mail: gajda@amu.edu.pl

Sebastian Petersen, Fachbereich 10, Universität Kassel, Wilhelmshöher Allee 73, 34121 Kassel, Germany
e-mail: petersen@mathematik.uni-kassel.de

Eingegangen 15. Oktober 2013, in revidierter Fassung 16. Februar 2015