Math. Nachr., 1 – 18 (2013) / **DOI** 10.1002/mana.201200217

Abelian varieties over finitely generated fields and the conjecture of Geyer and Jarden on torsion

Sara Arias-de-Reyna[*1], **Wojciech Gajda**[**2], and **Sebastian Petersen**[***3]

[1] Mathematics Research Unit, University of Luxembourg, L-1359 Luxembourg
[2] Department of Mathematics, Adam Mickiewicz University, 61614 Poznań, Poland
[3] Fachbereich Mathematik, Universität Kassel, 34132 Kassel, Germany

Received 29 July 2012, revised 20 September 2012, accepted 13 February 2013
Published online XXXX

Key words Abelian variety, Galois representation, Haar measure
MSC (2010) 11E30, 11G10, 14K15

In this paper we prove the Geyer-Jarden conjecture on the torsion part of the Mordell-Weil group for a large class of abelian varieties defined over finitely generated fields of arbitrary characteristic. The class consists of all abelian varieties with *big monodromy*, i.e., such that the image of Galois representation on ℓ-torsion points, for almost all primes ℓ, contains the full symplectic group.

1 Introduction

Let A be a polarized abelian variety defined over a finitely generated field K. Denote by $\widetilde{K}$ (respectively, K_{sep}) the algebraic (resp., separable) closure of K. It is well known that the Mordell-Weil group $A(K)$ is a finitely generated $\mathbb{Z}$-module. On the other hand $A(\widetilde{K})$ is a divisible group with an infinite torsion part $A(\widetilde{K})_{\mathrm{tor}}$ and $A(\widetilde{K})$ has infinite rank, unless K is algebraic over a finite field. Hence, it is of fundamental interest to study the structure of the groups $A(\Omega)$ for infinite algebraic extensions Ω/K smaller than $\widetilde{K}$. For example, Ribet in [18] and Zarhin in [24] considered the question of finiteness of $A(K_{\mathrm{ab}})_{\mathrm{tor}}$, where K_{ab} is the maximal abelian extension of K.

We denote by $G_K := G(K_{\mathrm{sep}}/K)$ the absolute Galois group of K. For a positive integer e and for $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_e)$ in the group $G_K^e = G_K \times G_K \times \cdots \times G_K$, we denote by $K_{\mathrm{sep}}(\sigma)$ the subfield in K_{sep} fixed by $\sigma_1, \sigma_2, \ldots, \sigma_e$. There exists a substantial literature on arithmetic properties of the fields $K_{\mathrm{sep}}(\sigma)$. In particular, the Mordell-Weil groups $A(K_{\mathrm{sep}}(\sigma))$ have been already studied, e.g., Larsen formulated a conjecture in [15] on the rank of $A(K_{\mathrm{sep}}(\sigma))$ (cf. [12], [7] for results supporting the conjecture of Larsen).

In this paper we consider the torsion part of the groups $A(K_{\mathrm{sep}}(\sigma))$. In order to recall the conjecture which is mentioned in the title, we agree to say that a property $\mathcal{A}(\sigma)$ *holds for almost all* $\sigma \in G_K^e$, if $\mathcal{A}(\sigma)$ holds for all $\sigma \in G_K^e$, except for a set of measure zero with respect to the (unique) normalized Haar measure on the compact group G_K^e. In [5] Geyer and Jarden proposed the following conjecture on the torsion part of $A(K_{\mathrm{sep}}(\sigma))$.

Conjecture of Geyer and Jarden Let K be a finitely generated field. Let A be an abelian variety defined over K.

(a) For almost all $\sigma \in G_K$ there are infinitely many prime numbers ℓ such that the group $A(K_{\mathrm{sep}}(\sigma))[\ell]$ of ℓ-division points is nonzero.

(b) Let $e \geq 2$. For almost all $\sigma \in G_K^e$ there are only finitely many prime numbers ℓ such that the group $A(K_{\mathrm{sep}}(\sigma))[\ell]$ of ℓ-division points is nonzero.

* e-mail: sara.ariasdereyna@uni.lu, Phone: +352 46 66 44 6269, Fax: xxxx Q1
** e-mail: gajda@amu.edu.pl, Phone: +48 61 829 5503, Fax: xxxx
*** Corresponding author: e-mail: sebastian.petersen@unibw.de, Phone: +49 561 804 4650, Fax: xxxx

It is known due to the work of Jacobson and Jarden [13] that for all $e \geq 1$, almost all $\sigma \in G_K^e$ and all primes ℓ the group $A(K_{\mathrm{sep}}(\sigma))[\ell^\infty]$ is finite. This was formerly part (c) of the conjecture. Moreover the conjecture is known for elliptic curves [5]. Part (b) holds true provided $\mathrm{char}(K) = 0$ (see [13]). In a very recent preprint Zywina proves part (a) in the special case where K is a number field (cf. [25]), stengthening results of Geyer and Jarden [6].

As for today, for an abelian variety A of dimension ≥ 2 defined over a finitely generated field of positive characteristic, parts (a) and (b) of the Conjecture of Geyer and Jarden are open and part (a) is open over a finitely generated transcendental extension of $\mathbb{Q}$.

In this paper we prove the Conjecture of Geyer and Jarden for abelian varieties with big monodromy. To formulate our main result we need some notation. Let $\ell \neq \mathrm{char}(K)$ be a prime number. We denote by $\rho_{A[\ell]} \colon G_K \to \mathrm{Aut}(A[\ell])$ the Galois representation attached to the action of G_K on the ℓ-torsion points of A. We define $\mathcal{M}_K(A[\ell]) := \rho_{A[\ell]}(G_K)$ and call this group *the mod-ℓ monodromy group of A/K*. We fix a polarization and denote by $e_\ell \colon A[\ell] \times A[\ell] \to \mu_\ell$ the corresponding Weil pairing. Then $\mathcal{M}_K(A[\ell])$ is a subgroup of the group of symplectic similitudes $\mathrm{GSp}(A[\ell], e_\ell)$ of the Weil pairing. We will say that *A/K has big monodromy* if there exists a constant ℓ_0 such that $\mathcal{M}_K(A[\ell])$ contains the symplectic group $\mathrm{Sp}(A[\ell], e_\ell)$, for every prime number $\ell \geq \ell_0$. Note that the property of having big monodromy does not depend on the choice of the polarization, cf. Proposition 3.6 below.

The main result of our paper is the following

Main Theorem [Cf. Thm. 4.1, Thm. 7.1.] *Let K be a finitely generated field and A/K an abelian variety with big monodromy. Then the Conjecture of Geyer and Jarden holds true for A/K.*

Surprisingly enough, the most difficult case to prove is Part (a) of the Conjecture for abelian varieties with big monodromy, when $\mathrm{char}(K) > 0$. The method of our proof relies in this case on the Borel-Cantelli Lemma of measure theory and on a delicate counting argument in the group $\mathrm{Sp}_{2g}(\mathbb{F}_\ell)$ which was modeled after a construction of subsets $S'(\ell)$ in $\mathrm{SL}_2(\mathbb{F}_\ell)$ in Section 3 of the classical paper [5] of Geyer and Jarden.

It is interesting to combine the main theorem with existing computations of monodromy groups for certain families of abelian varieties. We offer a result of this type in Corollary 7.2 below, thereby providing the reader with many examples of abelian varieties for which the conjecture of Geyer and Jarden is true.

2 Notation and background material

In this section we fix notation and gather some background material on Galois representations that is important for the rest of this paper.

If K is a field, then we denote by K_{sep} $\big(\text{resp. } \widetilde{K}\big)$ the separable (resp. algebraic) closure of K and by $G_K = G(K_{\mathrm{sep}}/K)$ its absolute Galois group. If G is a profinite (hence compact) group, then it has a unique normalized Haar measure μ_G. The expression "assertion $\mathcal{A}(\sigma)$ holds for almost all $\sigma \in G$" means "assertion $\mathcal{A}(\sigma)$ holds true for all σ outside a zero set with respect to μ_G". A finitely generated field is by definition a field which is finitely generated over its prime field. Let X be a scheme of finite type over a field K. For a geometric point $P \in X\big(\widetilde{K}\big)$ we denote by $K(P) \subset \widetilde{K}$ the residue field at P.

For $n \in \mathbb{N}$ coprime to $\mathrm{char}(K)$, we let $A[n]$ be the group of n-torsion points in $A\big(\widetilde{K}\big)$ and define $A[n^\infty] = \bigcup_{i=1}^\infty A[n^i]$. For a prime $\ell \neq \mathrm{char}(K)$ we denote by $T_\ell(A) = \varprojlim_{i \in \mathbb{N}} A[\ell^i]$ the ℓ-adic Tate module of A. Then $A[n]$, $A[n^\infty]$ and $T_\ell(A)$ are G_K-modules in a natural way.

If M is a G_K-module (for example $M = \mu_n$ or $M = A[n]$ where A/K is an abelian variety), then we shall denote the corresponding representation of the Galois group G_K by

$$\rho_M : G_K \longrightarrow \mathrm{Aut}(M)$$

and define $\mathcal{M}_K(M) := \rho_M(G_K)$. We define $K(M) := K_{\mathrm{sep}}^{\ker(\rho_M)}$ to be the fixed field in K_{sep} of the kernel of ρ_M. Then $K(M)/K$ is a Galois extension and $G(K(M)/K) \cong \mathcal{M}_K(M)$. For every algebraic extension L/K we define $\mathcal{M}_L(M) := \rho_M(G_L)$.

Let R be a commutative ring with 1 (usually $R = \mathbb{F}_\ell$ or $R = \mathbb{Z}_\ell$ or $R = \mathbb{Z}/n\mathbb{Z}$) and M a finitely generated free R-module equipped with an alternating bilinear pairing $e \colon M \times M \to R'$ into a free R-module R' of rank 1 (which is a multiplicatively written R-module in our setting below). We call such a pairing *perfect* provided the associated map

$$M \longrightarrow \mathrm{Hom}(M, R'), \quad x \longmapsto (y \mapsto e(x, y))$$

is bijective. We denote by

$$\mathrm{Sp}(M, e) = \{f \in \mathrm{Aut}_R(M) \mid \forall x, y \in M : e(f(x), f(y)) = e(x, y)\}$$

the corresponding symplectic group and by

$$\mathrm{GSp}(M, e) = \{f \in \mathrm{Aut}_R(M) \mid \exists \varepsilon \in R^\times : \forall x, y \in M : e(f(x), f(y)) = \varepsilon e(x, y)\}$$

the corresponding group of symplectic similitudes. Assume now that e is perfect. For $f \in \mathrm{GSp}(M, e)$ there is then even a unique value $\varepsilon(f) \in R^\times$ such that $e(f(x), f(y)) = \varepsilon(f)e(x, y)$ for all $x, y \in M$ and we call $\varepsilon(f)$ the *multiplicator* of f. The map

$$\mathrm{GSp}(M, e) \longrightarrow R^\times, \quad f \longmapsto \varepsilon(f)$$

is a homomorphism (cf. [2, Chap. 9, Paragraph 6, no. 5, p. 99]) which is called the *multiplicator map*.

Let n be an integer coprime to $\mathrm{char}(K)$ and ℓ be a prime different from $\mathrm{char}(K)$. We define the G_K-module $\mathbb{Z}_\ell(1)$ by

$$\mathbb{Z}_\ell(1) = \varprojlim_{j \in \mathbb{N}} \mu_{\ell^j} .$$

Let A/K be an abelian variety. We denote by $A^\vee$ the dual abelian variety and let $e_n : A[n] \times A^\vee[n] \to \mu_n$ and $e_{\ell^\infty} : T_\ell A \times T_\ell A^\vee \to \mathbb{Z}_\ell(1)$ be the corresponding Weil pairings (cf. [17, Chap. 16]). Choose a polarization $\lambda\colon A \to A^\vee$. (This is possible, cf. [3, Example 2.2, p. 8].) Consider the Weil pairings $e_n^\lambda : A[n] \times A[n] \to \mu_n$ and $e_{\ell^\infty}^\lambda : T_\ell A \times T_\ell A \to \mathbb{Z}_\ell(1)$ defined by $e_n^\lambda = e_n \circ (\mathrm{Id} \times \lambda)$ and by $e_{\ell^\infty}^\lambda = e_{\ell^\infty} \circ (\mathrm{Id} \times T_\ell(\lambda))$. If ℓ does not divide $\deg(\lambda)$ and if n is coprime to $\deg(\lambda)$, then e_n^λ and $e_{\ell^\infty}^\lambda$ are perfect, alternating, G_K-equivariant pairings (cf. [17, Chap. 16]). Hence we have representations

$$\rho_{A[n]} : G_K \longrightarrow \mathrm{GSp}\big(A[n], e_n^\lambda\big),$$

$$\rho_{T_\ell A} : G_K \longrightarrow \mathrm{GSp}\big(T_\ell A, e_{\ell^\infty}^\lambda\big),$$

and $\mathcal{M}_L(A[n]) = \rho_{A[n]}(G_L) \subset \mathrm{GSp}\big(A[n]), e_n^\lambda\big)$ and $\mathcal{M}_L(T_\ell A) = \rho_{T_\ell A}(G_L) \subset \mathrm{GSp}\big(T_\ell A, e_{\ell^\infty}^\lambda\big)$ for every algebraic extension L/K. The representations induce isomorphisms $G(L(A[n])/L) \cong \mathcal{M}_L(A[n])$ and $G(L(A[\ell^\infty]/L) \cong \mathcal{M}_L(T_\ell A)$. Note that $\mathcal{M}_L(T_\ell A) \to \mathcal{M}_L(A[\ell^i])$ is surjective (because $G(L(A[\ell^\infty])/L) \to G(L(A[\ell^i])/L)$ is surjective) for every integer i.

We shall say that an abelian variety (A, λ) over a field K has *big monodromy*, if there is a constant $\ell_0 > \max(\mathrm{char}(K), \deg(\lambda))$ such that $\mathcal{M}_K(A[\ell]) \supset \mathrm{Sp}\big(A[\ell], e_\ell^\lambda\big)$ for every prime number $\ell \geq \ell_0$. We will prove in Proposition 3.6 that the property of having big monodromy is independent of the choice of the polarization.

3 Properties of abelian varieties with big monodromy

Let (A, λ) be a polarized abelian variety with big monodromy over a finitely generated field K. Then it holds that $\mathrm{Sp}\big(A[\ell], e_\ell^\lambda\big) \subset \mathcal{M}_K(A[\ell])$ for sufficiently large primes ℓ. In this section we determine $\mathcal{M}_K(A[n])$ completely for every “sufficiently large” integer n. The main result (cf. Proposition 3.4 below) is due to Serre in the number field case, and the general case requires only a slight adaption of Serre’s line of reasoning. However, as the final outcome is somewhat different in positive characteristic, we do include the details. Proposition 3.4 will be crucial for our results on the Conjecture of Geyer and Jarden.

Remark 3.1 Let K be a field and (A, λ) a polarized abelian variety over K. Let n be an integer coprime to $\deg(\lambda)$.

(a) If L/K is a *Galois* extension, then $\mathcal{M}_L(A[n])$ is a *normal* subgroup of $\mathcal{M}_K(A[n])$ and the quotient group $\mathcal{M}_K(A[n])/\mathcal{M}_L(A[n])$ is isomorphic to $G(K(A[n]) \cap L/K)$.

(b) Define $K_n := K(A[n])$ and denote by $\overline{\rho}_{A[n]}\colon G(K_n/K) \to \mathrm{GSp}\big(A[n], e_n^\lambda\big)$ (resp. $\overline{\rho}_{\mu[n]}\colon G(K(\mu_n)/K) \to (\mathbb{Z}/n\mathbb{Z})^\times)$) the monomorphism induced by $\rho_{A[n]}$ (resp. by the cyclotomic character ρ_{μ_n}). Recall that $\varepsilon\colon \mathrm{GSp}\big(A[n], e_n^\lambda\big) \to (\mathbb{Z}/n\mathbb{Z})^\times$ is the multiplicator map.
Then $K(\mu_n) \subset K_n$, $\mathcal{M}_{K(\mu_n)}(A[n]) \subset \mathrm{Sp}\big(A[n], e_n^\lambda\big)$ and there is a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G(K_n/K(\mu_n)) & \longrightarrow & G(K_n/K) & \longrightarrow & G(K(\mu_n)/K) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \overline{\rho}_{A[n]}} & & \downarrow{\scriptstyle \overline{\rho}_{\mu_n}} & & \\
1 & \longrightarrow & \mathrm{Sp}\big(A[n], e_n^\lambda\big) & \longrightarrow & \mathrm{GSp}\big(A[n], e_n^\lambda\big) & \xrightarrow{\varepsilon} & (\mathbb{Z}/n\mathbb{Z})^\times & \longrightarrow & 1
\end{array}$$

with exact rows and injective vertical maps.

(c) If $\mathrm{Sp}\big(A[n], e_n^\lambda\big) \subset \mathrm{im}\big(\rho_{A[n]}\big)$, then the left-hand vertical map is an isomorphism and $\mathcal{M}_{K(\mu_n)}(A[n]) = \mathrm{Sp}\big(A[n], e_n^\lambda\big)$.

P r o o f. Part (a). If L/K is Galois, then G_L is normal in G_K, and hence $\mathcal{M}_L(A[n]) = \rho_{A[n]}(G_L)$ is normal in $\mathcal{M}_K(A[n]) = \rho_{A[n]}(G_K)$. The second isomorphism theorem implies that $\mathcal{M}_K(A[n])/\mathcal{M}_L(A[n])$ is isomorphic to the group $G_K / \ker\big(\rho_{A[n]}\big) \cdot G_L = \mathrm{Gal}(K(A[n]) \cap L/K)$.

Part (b). Denote by $\zeta \in \mu_n$ a primitive n-th root of unity. Then there exist $P, Q \in A[n]$ such that $e_n^\lambda(P,Q) = \zeta$, because e_n^λ is a perfect pairing. For all $\sigma \in G_{K_n}$ we have

$$\sigma(\zeta) = \sigma\big(e_n^\lambda(P,Q)\big) = e_n^\lambda\big(\rho_{A[n]}(\sigma)(P), \rho_{A[n]}(\sigma)(Q)\big) = e_n^\lambda(P,Q) = \zeta$$

by the G_K-equivariance of the Weil pairing. It follows that $G_{K_n} \subset G_{K(\mu_n)}$ and $K(\mu_n) \subset K_n = K(A[n])$. We have thus established the upper exact sequence. Furthermore, again by the G_K-equivariance of the Weil pairing, we have

$$e_n^\lambda(\rho_{A[n]}(\sigma)(P), \rho_{A[n]}(\sigma)(Q)) = \sigma\big(e_n^\lambda(P,Q)\big) = e_n^\lambda(P,Q)^{\rho_{\mu_n}(\sigma)}$$

for all $P, Q \in A[n]$ and all $\sigma \in G_K$. This implies that the right rectangle in the diagram is commutative and that $\mathcal{M}_{K(\mu_n)}(A[n]) \subset \mathrm{Sp}\big(A[n], e_n^\lambda\big)$. We define the right vertical arrow to be the restriction of $\overline{\rho}_{A[n]}$ to $G(K_n/K(\mu_n))$ to the kernel of the upper sequence. Then the left rectangle in the diagram is commutative by construction. Finally the injectivity of the middle arrow implies that the left vertical arrow is injective.

Part (c). Assume that $\mathrm{Sp}\big(A[n], e_n^\lambda\big) \subset \mathrm{im}\big(\rho_{A[n]}\big)$ and let $f \in \mathrm{Sp}\big(A[n], e_n^\lambda\big)$. Then there exists $\sigma \in G(K_n/K)$ such that $\overline{\rho}_{A[n]}(\sigma) = f$. Then $\overline{\rho}_{\mu_n}(\sigma|K(\mu_n)) = \varepsilon(f) = 1$, hence $\sigma|K(\mu_n) = \mathrm{Id}$, because $\overline{\rho}_{\mu_n}$ is injective. Thus $\sigma \in G_{K(\mu_n)}$ and the assertion follows from that. □

Proposition 3.2 *Let K be a field and (A,λ) a polarized abelian variety over K with big monodromy. Let L/K be an abelian Galois extension with $L \supset \mu_\infty$. Then there is a constant $\ell_0 > \max(\mathrm{char}(K), \deg(\lambda))$ with the following properties.*

(a) $\mathcal{M}_L(T_\ell A) = \mathrm{Sp}\big(T_\ell A, e_{\ell^\infty}^\lambda\big)$ *for all primes* $\ell \geq \ell_0$.

(b) *Let c be the product of all prime numbers $\leq \ell_0$. Then $\mathcal{M}_L(A[n]) = \mathrm{Sp}\big(A[n], e_n^\lambda\big)$ for every integer n which is coprime to c.*

P r o o f. Part (a). There is a constant $\ell_0 > \max(\mathrm{char}(K), \deg(\lambda), 5)$ such that $\mathcal{M}_K(A[\ell]) \supset \mathrm{Sp}\big(A[\ell], e_\ell^\lambda\big)$ for all primes $\ell \geq \ell_0$, because A has big monodromy. Let $\ell \geq \ell_0$ be a prime. Then

$$\mathrm{Gal}(K(A[\ell])/K(\mu_\ell)) \cong \mathcal{M}_{K(\mu_\ell)}(A[\ell]) = \mathrm{Sp}\big(A[\ell], e_\ell^\lambda\big)$$

by Remark 3.1, part (c).

The group $\mathrm{Sp}\big(A[\ell], e_\ell^\lambda\big)$ is perfect, because $\ell \geq 5$ (cf. [22, Thm. 8.7]). As $L/K(\mu_\ell)$ is an abelian Galois extension, $\mathcal{M}_L(A[\ell])$ is a normal subgroup of the perfect group $\mathcal{M}_{K(\mu_\ell)}(A[\ell])$ and the quotient group

$\mathcal{M}_{K(\mu_\ell)}(A[\ell])/\mathcal{M}_L(A[\ell])$ is isomorphic to a subquotient of $G(L/K)$ (cf. Remark 3.1, part a), hence abelian. This implies that

$$\mathcal{M}_L(A[\ell]) = \mathcal{M}_{K(\mu_\ell)}(A[\ell]) = \mathrm{Sp}\big(A[\ell], e_\ell^\lambda\big).$$

Denote by $p\colon \mathrm{Sp}\big(T_\ell A, e_{\ell^\infty}^\lambda\big) \to \mathrm{Sp}\big(A[\ell], e_\ell^\lambda\big)$ the canonical projection. Then $\mathcal{M}_L(T_\ell A)$ is a closed subgroup of $\mathrm{Sp}\big(T_\ell A, e_{\ell^\infty}^\lambda\big)$ with

$$p(\mathcal{M}_L(T_\ell A)) = \mathcal{M}_L(A[\ell]) = \mathrm{Sp}\big(A[\ell], e_\ell^\lambda\big).$$

Hence $\mathcal{M}_L(T_\ell A) = \mathrm{Sp}\big(T_\ell A, e_{\ell^\infty}^\lambda\big)$ (cf. [14, Prop. 2.6], [23, Thm. B]).

Part (b). Consider the map

$$\rho\colon G_L \to \prod_{\ell \ge \ell_0} \mathcal{M}_L(T_\ell A) = \prod_{\ell \ge \ell_0} \mathrm{Sp}\big(T_\ell A, e_{\ell^\infty}^\lambda\big)$$

induced by the representations $\rho_{T_\ell A}$ and denote by $X := \rho(G_L)$ its image. Then X is a closed subgroup of $\prod_{\ell \ge \ell_0} \mathrm{Sp}\big(T_\ell A, e_{\ell^\infty}^\lambda\big)$. If pr_ℓ denotes the ℓ-th projection of the product, then $\mathrm{pr}_\ell(X) = \mathrm{Sp}\big(T_\ell A, e_{\ell^\infty}^\lambda\big)$. Hence [21, Section 7, Lemme 2] implies that $X = \prod_{\ell \ge \ell_0} \mathrm{Sp}\big(T_\ell A, e_{\ell^\infty}^\lambda\big)$, i.e. that ρ is *surjective*.

Let c be the product of all prime numbers $\le \ell_0$. Let n be an integer coprime to c. Then $n = \prod_{\ell | n \text{ prime}} \ell^{v_\ell}$ for certain integers $v_\ell \ge 1$. The canonical map $r\colon \mathcal{M}_L(A[n]) \to \prod_{\ell | n \text{ prime}} \mathcal{M}_L(A[\ell^{v_\ell}])$ is injective. Consider the diagram

$$\begin{array}{ccccc}
G_L & \xrightarrow{\rho'} & \prod_{\ell|n} \mathcal{M}_L(T_\ell A) & = & \prod_{\ell|n} \mathrm{Sp}\big(T_\ell A, e_{\ell^\infty}^\lambda\big) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}_L(A[n]) & \xhookrightarrow{r} & \prod_{\ell|n} \mathcal{M}_L(A[\ell^{v_\ell}]) & \hookrightarrow & \prod_{\ell|n} \mathrm{Sp}\big(A[\ell^{v_\ell}], e_{\ell^{v_\ell}}^\lambda\big).
\end{array}$$

The vertical maps are surjective. The horizontal map ρ' is surjective as well, because ρ is surjective. This implies, that the lower horizontal map

$$\mathcal{M}_L(A[n]) \to \prod_{\ell|n} \mathrm{Sp}\big(A[\ell^{v_\ell}], e_{\ell^{v_\ell}}^\lambda\big)$$

is in fact bijective. It follows from the Chinese Remainder Theorem that the canonical map

$$\prod_{\ell|n} \mathrm{Sp}\big(A[\ell^{v_\ell}], e_{\ell^{v_\ell}}^\lambda\big) \to \mathrm{Sp}\big(A[n], e_n^\lambda\big)$$

is bijective as well. Assertion (b) follows from that. □

Corollary 3.3 *Let K be a field and (A, λ) a polarized abelian variety over K with big monodromy. Then there is a constant c coprime to $\deg(\lambda)$ and to $\mathrm{char}(K)$, if $\mathrm{char}(K)$ is positive, with the following property: $\mathcal{M}_K(A[n]) \supset \mathrm{Sp}\big(A[n], e_n^\lambda\big)$ for every integer n coprime to c.*

Proof. Let $L = K_{\mathrm{ab}}$ be the maximal abelian extension. Then there is a constant c as above, such that $\mathcal{M}_L(A[n]) = \mathrm{Sp}\big(A[n], e_n^\lambda\big)$ for every n coprime to c by Proposition 3.2. Furthermore $\mathcal{M}_L(A[n]) \subset \mathcal{M}_K(A[n])$ by Remark 3.1, part (a). □

Let K be a field and (A, λ) a polarized abelian variety over K with big monodromy. There is a constant c (divisible by $\deg(\lambda)$ and by $\mathrm{char}(K)$, if $\mathrm{char}(K) \neq 0$) such that

$$\mathrm{Sp}\big(A[n], e_n^\lambda\big) \subset \mathcal{M}_K(A[n]) \subset \mathrm{GSp}\big(A[n], e_n^\lambda\big)$$

for all $n \in \mathbb{N}$ coprime to c (cf. Corollary 3.3). From the diagram in Remark 3.1 one sees that

$$\mathcal{M}_K(A[n]) = \{f \in \mathrm{GSp}(A[n], e_n^\lambda) \mid \varepsilon(f) \in \mathrm{im}(\rho_{\mu_n})\}.$$

for all $n \in \mathbb{N}$ coprime to c. If K is finitely generated, then one can determine $\mathrm{im}(\rho_{\mu_n})$ and $\mathcal{M}_K(A[n])$ completely.

Assume from now on that K is finitely generated. Then the image of the cyclotomic character involved above has a well-known explicit description. Denote by F the algebraic closure of the prime field of K in K and define $q := q(K) := |F| \in \mathbb{N} \cup \{\infty\}$. Then, after possibly replacing c by a larger constant, we have

$$\mathrm{im}(\rho_{\mu_n}) = \begin{cases} \langle \overline{q} \rangle, & \mathrm{char}(K) \neq 0, \\ (\mathbb{Z}/n\mathbb{Z})^\times, & \mathrm{char}(K) = 0, \end{cases}$$

for all $n \in \mathbb{N}$ coprime to c (cf. [16, Thm. 2.47(ii)]). Here $\langle \overline{q} \rangle$ is the subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$ generated by the residue class $\overline{q}$ of q modulo n, provided q is finite. If q is finite, then we define

$$\mathrm{GSp}^{(q)}(A[n], e_n^\lambda) = \{f \in \mathrm{GSp}(A[n], e_n^\lambda) \mid \varepsilon(f) \in \langle \overline{q} \rangle\}.$$

Finally we put $\mathrm{GSp}^{(\infty)}(A[n], e_n^\lambda) = \mathrm{GSp}(A[n], e_n^\lambda)$. We have shown:

Proposition 3.4 *Let K be a finitely generated field and (A, λ) a polarized abelian variety over K with big monodromy. Let $q = q(K)$. Then there is a constant c (divisible by $\deg(\lambda)$ and by $\mathrm{char}(K)$, if $\mathrm{char}(K) \neq 0$) such that $\mathcal{M}_K(A[n]) = \mathrm{GSp}^{(q)}(A[n], e_n^\lambda)$ for all $n \in \mathbb{N}$ coprime to c.*

We shall now prove that the notion of big monodromy does not depend on the choice of the polarization. For this we need the following lemma.

Lemma 3.5 *Let T be a finitely generated free $\mathbb{Z}_\ell$-module and $e\colon T \times T \to \mathbb{Z}_\ell$ a perfect alternating bilinear pairing. Then*

$$\{f \in \mathrm{End}_{\mathbb{Z}_\ell}(T) \mid\ f \circ g = g \circ f\ \forall g \in \mathrm{Sp}(T, e)\} = \mathbb{Z}_\ell \mathrm{Id}_T.$$

P r o o f. Let $f \in \mathrm{End}_{\mathbb{Z}_\ell}(T)$ and assume that $f \circ g = g \circ f$ for all $g \in \mathrm{Sp}(T, e)$. Note that for every $u \in T$ the automorphism $T_u : v \mapsto v + e(v, u)u$ lies in $\mathrm{Sp}(T, e)$ (cf. [8, Chap. 3, p. 23]). Then $f \circ T_u(v) = f(v) + e(v, u)f(u)$ and $T_u \circ f(v) = f(v) + e(f(v), u)u$. It follows that

$$e(v, u)f(u) = e(f(v), u)u \quad \text{for all} \quad u, v \in T.$$

Now choose an arbitrary $\mathbb{Z}_\ell$-basis $(u_1, \ldots, u_n)$ of T. For every index i there is a vector v_i such that $e(v_i, u_i) = 1$ and $e(v_i, u_j) = 0$ for all $i \neq j$, because the pairing e is perfect. It follows that $f(u_i) = e(f(v_i), u_i)u_i$ for all i. We put $\lambda_i := e(f(v_i), u_i)$ such that $f(u_i) = \lambda_i u_i$.

For $i \neq 1$ we have $e(v_1, u_1 + u_j) = 1$, hence $f(u_1 + u_j) = e(f(v_1), u_1 + u_j)(u_1 + u_j)$. We put $\lambda_{1,j} = e(f(v_1), u_1 + u_j)$ such that $f(u_1 + u_j) = \lambda_{1,j}(u_1 + u_j)$. Then on the one hand $f(u_1 + u_j) = \lambda_{1,j}u_1 + \lambda_{1,j}u_j$. On the other hand $f(u_1 + u_j) = f(u_1) + f(u_j) = \lambda_1 u_1 + \lambda_j u_j$. This implies $\lambda_1 = \lambda_{1,j} = \lambda_j$. Hence $f = \lambda_1 \mathrm{Id}_T$. □

Proposition 3.6 *Let K be a field and (A, λ) a polarized non-zero abelian variety over K with big monodromy.*

(a) $\mathrm{End}_K(A) = \mathbb{Z}$.

(b) *For every other polarization $\lambda'\colon A \to A^\vee$ there exist $a, b \in \mathbb{Z}$ such that $a\lambda = b\lambda'$ and $\mathrm{Sp}(A[n], e_n^\lambda) = \mathrm{Sp}(A[n], e_n^{\lambda'})$ for all n coprime to $ab\mathrm{char}(K)$.*

P r o o f. Part (a). Fix one large enough prime number $\ell \neq \mathrm{char}(K)$ such that $\mathrm{Sp}(T_\ell A, e_{\ell^\infty}^\lambda) \subset \mathcal{M}_K(A[\ell])$. This is possible because A has big monodromy by Proposition 3.2. The canonical morphism

$$i\colon \mathrm{End}_K(A) \otimes \mathbb{Z}_\ell \longrightarrow \mathrm{End}_{\mathbb{Z}_\ell}(T_\ell(A))$$

is injective and $\mathrm{End}_K(A)$ is a $\mathbb{Z}$-algebra which is finitely generated and free as a $\mathbb{Z}$-module (cf. [17, Lemma 11.2]). The image $\mathrm{im}(i)$ is contained in

$$\mathrm{End}_{\mathbb{Z}_\ell}(T_\ell A)^{G_K} = \{f \in \mathrm{End}_{\mathbb{Z}_\ell}(T_\ell A) \mid f \circ \rho_{T_\ell A}(\sigma) = \rho_{T_\ell A}(\sigma) \circ f \ \forall \sigma \in G_K\}$$

(If K is finitely generated, then $\mathrm{im}(i) = \mathrm{End}_{\mathbb{Z}_\ell}(T_\ell A)^{G_K}$ by a famous theorem of Faltings, but we do not make use of this deep theorem.) As $\rho_{T_\ell A}(G_K) \supset \mathrm{Sp}(T_\ell A, e^\lambda_{\ell^\infty})$, Lemma 3.5 implies $\mathrm{End}_{\mathbb{Z}_\ell}(T_\ell A)^{G_K} = \mathbb{Z}_\ell \mathrm{Id}$. It follows that

$$\mathrm{rk}_{\mathbb{Z}_\ell}(\mathrm{End}_K(A) \otimes \mathbb{Z}_\ell) = 1.$$

Because $\mathrm{End}_K(A)$ is finitely generated and free, this implies $\mathrm{rk}_{\mathbb{Z}}(\mathrm{End}_K(A)) = 1$ and $\mathrm{End}_K(A) = \mathbb{Z}$.

Part (b). The polarization $\lambda\colon A \to A^\vee$ is an isogeny. Hence there exists a polarization $\xi\colon A^\vee \to A$ and the homomorphism

$$j\colon \mathrm{Hom}_K(A, A^\vee) \longrightarrow \mathrm{End}(A), \quad f \longmapsto \xi \circ f$$

is injective. (If $f \in \ker(j)$, then $\xi \circ f = 0$, hence $\mathrm{im}(f) \subset \ker(\xi)$, and this implies $\mathrm{im}(f) = 0$ because $\mathrm{im}(f)$ is connected and $\ker(f)$ is finite.) Hence $\mathrm{Hom}_K(A, A^\vee)$ is a free $\mathbb{Z}$-module of rank 1. As $\lambda, \lambda' \in \mathrm{Hom}_K(A, A^\vee)$, we see that there are $a, b \in \mathbb{Z}$ such that $a\lambda = b\lambda'$. Now let $n \in \mathbb{N}$ be coprime to $ab\mathrm{char}(K)$. Then $ae^\lambda_n(P,Q) = be^{\lambda'}_n(P,Q)$ for all $P, Q \in A[n]$. Because the residue classes of a and b lie in $(\mathbb{Z}/n\mathbb{Z})^\times$, this implies $\mathrm{Sp}(A[n], e^\lambda_n) = \mathrm{Sp}(A[n], e^{\lambda'}_n)$. □

4 Proof of the Conjecture of Geyer and Jarden, part (b)

Let (A, λ) be a polarized abelian variety of dimension g over a field K. In this section we will use the notation $K_\ell := K(A[\ell])$ and $G_\ell := G(K_\ell/K)$ for every prime $\ell \neq \mathrm{char}(K)$. Our main result in this section is the following theorem.

Theorem 4.1 *If (A, λ) has big monodromy, then for all $e \geq 2$ and almost all $\sigma \in G^e_K$ (in the sense of the Haar measure) there are only finitely many primes ℓ such that $A(K_{\mathrm{sep}}(\sigma))[\ell] \neq 0$.*

The following Lemma 4.2 is due to Oskar Villareal (private communication). We thank him for his kind permission to include it into our manuscript. This section in to a large extent inspired by an unpublished note of him.

Lemma 4.2 *Assume that A has big monodromy. Then there is a constant ℓ_0 such that $[K(P) : K]^{-1} \leq [K_\ell : K]^{-\frac{1}{2g}}$ for all primes $\ell \geq \ell_0$ and all $P \in A[\ell] \smallsetminus \{0\}$, where $K(P)$ denotes the residue field of the point P.*

Proof. By assumption on A, there is a constant ℓ_0 such that $\mathrm{Sp}(A[\ell], e^\lambda_\ell) \subset \mathcal{M}_K(A[\ell])$ for all primes $\ell \geq \ell_0$. Let $\ell \geq \ell_0$ be a prime and $P \in A[\ell] \smallsetminus \{0\}$. Then the $\mathbb{F}_\ell$-vector space generated inside $A[\ell]$ by the orbit $X := \{f(P) \mid f \in \mathcal{M}_K(A[\ell])\}$ is the whole of $A[\ell]$, because $A[\ell]$ is a simple $\mathbb{F}_\ell[\mathrm{Sp}(A[\ell], e^\lambda_\ell)]$-module (cf. [11, Satz 9.15, p. 221]). Thus we can choose an $\mathbb{F}_\ell$-basis $(P_1, \ldots, P_{2g})$ of $A[\ell]$ with $P_1 = P$ in such a way that each $P_i \in X$. Then each P_i is conjugate to P under the action of G_K and $[K(P) : K] = [K(P_i) : K]$ for all i. The field K_ℓ is the composite field $K_\ell = K(P_1) \ldots K(P_{2g})$. It follows that

$$[K_\ell : K] \leq [K(P_1) : K] \ldots [K(P_{2g}) : K] = [K(P) : K]^{2g}.$$

The desired inequality follows from that. □

The following notation will be used in the sequel: For sequences $(x_n)_n$ and $(y_n)_n$ of positive real numbers we shall write $x_n \sim y_n$, provided the sequence $\left(\frac{x_n}{y_n}\right)_n$ converges to a positive real number. If $x_n \sim y_n$ and $\sum_{n=1}^\infty x_n < \infty$, then $\sum_{n=1}^\infty y_n < \infty$.

The proof of Theorem 4.1 will make heavy use of the following classical fact.

Lemma 4.3 (Borel-Cantelli, [4, 18.3.5].) *Let $(A_1, A_2, \ldots)$ be a sequence of measurable subsets of a profinite group G. Let*

$$A := \bigcap_{n=1}^\infty \bigcup_{i=n}^\infty A_i = \{x \in G \mid x \text{ belongs to infinitely many } A_i\}.$$

(a) *If* $\sum_{i=1}^{\infty} \mu_G(A_i) < \infty$, *then* $\mu_G(A) = 0$.

(b) *If* $\sum_{i=1}^{\infty} \mu_G(A_i) = \infty$ *and* $(A_i)_{i\in\mathbb{N}}$ *is a* μ_G*-independent sequence* $\big($*i.e. for every finite set* $I \subset \mathbb{N}$ *we have* $\mu_G\big(\bigcap_{i\in I} A_i\big) = \prod_{i\in I} \mu_G(A_i)\big)$, *then* $\mu_G(A) = 1$.

Proof of Theorem 4.1. Assume that A/K has big monodromy and let ℓ_0 be a constant as in the definition of the term "big monodromy". We may assume that $\ell_0 \geq \mathrm{char}(K)$. Let $e \geq 2$ and define

$$X_\ell := \{\sigma \in G_K^e \mid A(K_{\mathrm{sep}}(\sigma))[\ell] \neq 0\}$$

for every prime ℓ. Let μ be the normalized Haar measure on G_K^e. Theorem 4.1 follows from Claim 1 below, because Claim 1 together with the Borel-Cantelli Lemma 4.3 implies that

$$\bigcap_{n\in\mathbb{N}} \bigcup_{\ell\geq n \text{ prime}} X_\ell$$

has measure zero.

Claim 1. *The series* $\sum_{\ell \text{ prime}} \mu(X_\ell)$ *converges.*

Let $\ell \geq \ell_0$ be a prime number. Note that

$$X_\ell = \bigcup_{P\in A[\ell]\smallsetminus\{0\}} \big\{\sigma \in G_K^e \mid \sigma_i(P) = P \text{ for all } i\big\} = \bigcup_{P\in A[\ell]\smallsetminus\{0\}} G_{K(P)}^e.$$

Let $\mathbb{P}(A[\ell]) = (A[\ell] \smallsetminus \{0\})/\mathbb{F}_\ell^\times$ be the projective space of lines in the $\mathbb{F}_\ell$-vector space $A[\ell]$. It is a projective space of dimension $2g-1$. For $P \in A[\ell] \smallsetminus \{0\}$ we denote by $\overline{P} := \mathbb{F}_\ell^\times P$ the equivalence class of P in $\mathbb{P}(A[\ell])$. For $\overline{P} \in \mathbb{P}(A[\ell])$ and $P_1, P_2 \in \overline{P}$ there is an $a \in \mathbb{F}_\ell^\times$ such that $P_1 = aP_2$ and $P_2 = a^{-1}P_1$, and this implies $K(P_1) = K(P_2)$. It follows that we can write

$$X_\ell = \bigcup_{\overline{P}\in\mathbb{P}(A[\ell])} G_{K(P)}^e.$$

Hence

$$\mu(X_\ell) \leq \sum_{\overline{P}\in\mathbb{P}(A[\ell])} \mu\big(G_{K(P)}^e\big) = \sum_{\overline{P}\in\mathbb{P}(A[\ell])} [K(P):K]^{-e},$$

and Lemma 4.2 implies

$$\mu(X_\ell) \leq \sum_{\overline{P}\in\mathbb{P}(A[\ell])} [K_\ell:K]^{-e/2g} = \frac{\ell^{2g}-1}{\ell-1}[K_\ell:K]^{-e/2g} = \frac{\ell^{2g}-1}{\ell-1}|G_\ell|^{-e/2g}.$$

But G_ℓ contains $\mathrm{Sp}_{2g}(\mathbb{F}_\ell)$ and

$$s_\ell := \big|\mathrm{Sp}_{2g}(\mathbb{F}_\ell)\big| = \ell^{g^2}\prod_{i=1}^{g}\big(\ell^{2i}-1\big)$$

(cf. [22]). It is thus enough to prove the following

Claim 2. *The series* $\sum_{\ell\geq\ell_0 \text{ prime}} \frac{\ell^{2g}-1}{\ell-1} s_\ell^{-e/2g}$ *converges.*

But $s_\ell \sim \ell^{g^2+2+4+\cdots+2g} = \ell^{2g^2+g}$ and $\frac{\ell^{2g}-1}{\ell-1} \sim \ell^{2g-1}$, hence

$$\frac{\ell^{2g}-1}{\ell-1}s_\ell^{-e/2g} \sim \ell^{2g-1}\ell^{-e(g+\frac{1}{2})} = \ell^{(2-e)g-(1+\frac{e}{2})} \leq \ell^{-2},$$

because $e \geq 2$. Claim 2 follows from that. □

5 Special sets of symplectic matrices over $\mathbb{F}_\ell$

This section contains a construction of certain special sets of symplectic matrices (cf. Theorem 5 below) that will play a crucial role in the proof of part (a) of the Conjecture of Geyer and Jarden.

Let R be a commutative ring (usually $R = \mathbb{Z}/n\mathbb{Z}$ or $R = \mathbb{Z}_\ell$ in our applications). For $g \geq 2$ we consider the free R-module R^{2g} and denote by $(e_1, \dots, e_{2g})$ the standard basis of R^{2g}. We shall always identify a matrix $A \in \mathrm{GL}_{2g}(R)$ with the corresponding automorphism $x \mapsto Ax$ of R^{2g}. Let

$$J_g = \begin{pmatrix} J_1 & & & \\ & J_1 & & \\ & & \ddots & \\ & & & J_1 \end{pmatrix} \in \mathrm{GL}_{2g}(R) \quad \text{where} \quad J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then there is a perfect alternating bilinear pairing $e\colon R^{2g} \times R^{2g} \to R$ defined by $e(x, y) := x^t J_g y$. This pairing e is called the canonical symplectic pairing. Note that $e(e_i, e_{i+1}) = 1 = -e(e_{i+1}, e_i)$ and $e(e_i, e_j) = 0$ for all odd i and all $j \neq i + 1$. We define $\mathrm{Sp}_{2g}(R) = \mathrm{Sp}\big(R^{2g}, e\big)$ and $\mathrm{GSp}_{2g}(R) = \mathrm{GSp}\big(R^{2g}, e\big)$ (cf. Section 1). Recall from Section 2 that there is a homomorphism $\varepsilon\colon \mathrm{GSp}_{2g}(R) \to R^\times$, called the multiplicator map, such that $e(Ax, Ay) = \varepsilon(A)e(x, y)$ for all $x, y \in R^{2g}$ and all $A \in \mathrm{GSp}_{2g}(R)$. For $\lambda \in R^\times$ we define

$$\mathrm{GSp}_{2g}(R)[\lambda] := \big\{A \in \mathrm{GSp}_{2g}(\mathbb{Z}/n\mathbb{Z}) \mid \varepsilon(A) = \lambda\big\}.$$

Now consider the special case $R = \mathbb{Z}/n\mathbb{Z}$. If q is a prime power coprime to n, then we denote by $\overline{q}$ its residue class in $(\mathbb{Z}/n\mathbb{Z})^\times$ and by $\mathrm{ord}_n(q) = |\langle\overline{q}\rangle|$ the order of $\overline{q}$ as element of the group $(\mathbb{Z}/n\mathbb{Z})^\times$. Recall from Section 3 that

$$\mathrm{GSp}_{2g}^{(q)}(\mathbb{Z}/n\mathbb{Z}) = \big\{A \in \mathrm{GSp}_{2g}(\mathbb{Z}/n\mathbb{Z}) \mid \varepsilon(A) \in \langle\overline{q}\rangle\big\}$$

and $\mathrm{GSp}_{2g}^{(\infty)}(\mathbb{Z}/n\mathbb{Z}) = \mathrm{GSp}_{2g}(\mathbb{Z}/n\mathbb{Z})$.

For the rest of this section we specialize to the case $R = \mathbb{F}_\ell$ and put $V := \mathbb{F}_\ell^{2g}$. For $u \in V$ and $\beta \in \mathbb{F}_\ell$ consider the automorphism

$$T_u[\beta]\colon v \longmapsto v + \beta e(v, u)u$$

of V. Then $T_u[\beta]$ is a transvection contained in $\mathrm{Sp}_{2g}(\mathbb{F}_\ell)$ and furthermore the map

$$(\mathbb{F}_\ell, +) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{F}_\ell), \quad \beta \longmapsto T_u[\beta]$$

is a homomorphism.

We begin with two elementary lemmas that will be essential for Definition 5.3.

Lemma 5.1 *Let ℓ be a prime number. For each $\lambda \in \mathbb{F}_\ell^\times$, the matrices of $\mathrm{GSp}_{2g}(\mathbb{F}_\ell)[\lambda]$ that fix the vector e_1 are of the form*

$$\left(\begin{array}{c|c|ccc} 1 & d & b_1 & b_2 & \dots \\ \hline 0 & \lambda & 0 & 0 & \dots \\ \hline 0 & d_1 & & & \\ \vdots & \vdots & & B & \\ \vdots & \vdots & & & \end{array}\right) \tag{5.1}$$

with $B = (b_{ij})_{i,j=1,\dots,2g-2} \in \mathrm{GSp}_{2g-2}(\mathbb{F}_\ell)[\lambda]$, $d, d_1, \dots, d_{2g-2} \in \mathbb{F}_\ell$ and

$$b_k = \frac{1}{\lambda}\left(\sum_{j=1}^{g-1}\big(d_{2j-1}b_{2j,k} - d_{2j}b_{2j-1,k}\big)\right) \in \mathbb{F}_\ell \quad \textit{for each} \quad k = 1, \dots, 2g-2. \tag{5.2}$$

P r o o f. Let $A \in \mathrm{GSp}_{2g}(\mathbb{F}_\ell)[\lambda]$ be such that $Ae_1 = e_1$. Let us write the matrix of A with respect to the symplectic basis $\{e_1, e_2, \ldots, e_{2g-1}, e_{2g}\}$. For each $k = 3, \ldots, 2g$ we have $e(e_1, e_k) = 0$, so $e(e_1, Ae_k) = 0$. Therefore we can write the matrix A as

$$A = \left(\begin{array}{c|c|ccc} 1 & d & b_1 & b_2 & \ldots \\ \hline 0 & d' & 0 & 0 & \ldots \\ \hline 0 & d_1 & & & \\ \vdots & \vdots & & B & \\ \vdots & \vdots & & & \end{array}\right)$$

where in the second row we get all entries zero save the $(2,2)$-th. Moreover, since $e(e_1, e_2) = 1$, we get that $e(e_1, Ae_2) = e(Ae_1, Ae_2) = \lambda e(e_1, e_2) = \lambda$, that is to say, $d' = \lambda$. Furthermore, we have that $e(e_2, e_k) = 0$ for all $k = 3, \ldots, 2g$, hence $e(Ae_2, Ae_k) = 0$. This gives rise to the Equations (5.2). Denote by e' the canonical symplectic pairing on $\mathbb{F}_\ell^{2g-2}$ and by $(e'_1, \ldots, e'_{2g-2})$ the standard basis of $\mathbb{F}_\ell^{2g-2}$. Then $e(Ae_i, Ae_j) = e'(Be'_{i-2}, Be'_{j-2})$ for $i, j \geq 3$. Hence the fact that $A \in \mathrm{GSp}_{2g}(\mathbb{F}_\ell)[\lambda]$ implies that $B \in \mathrm{GSp}_{2g-2}(\mathbb{F}_\ell)[\lambda]$. This proves that the conditions in the lemma are necessary.

We prove that they are also sufficient. Let A be a matrix satisfying conditions (1) and (2) of the lemma. Then $Ae_1 = e_1$ because the first column of A is e_1. Furthermore $e(Ae_1, Ae_2) = \lambda = \lambda e(e_1, e_2)$ and $e(Ae_1, Ae_k) = 0 = \lambda e(Ae_1, Ae_k)$ for all $k \geq 3$. For $k \geq 3$ we have

$$e(Ae_2, Ae_k) = -\lambda b_k + \left(\sum_{j=1}^{g-1} \left(d_{2j-1} b_{2j,k} - d_{2j} b_{2j-1,k}\right)\right) = 0 = \lambda e(e_2, e_k)$$

because of the Equations (5.2). Finally

$$e\big(Ae_i, Ae_j\big) = e'\big(Be'_{i-2}, Be'_{j-2}\big) = \lambda e'\big(e'_{i-2}, e'_{j-2}\big) = \lambda e\big(e_i, e_j\big)$$

for all $3 \leq i < j$, because $B \in \mathrm{GSp}_{2g-2}(\mathbb{F}_\ell)[\lambda]$. Altogether we see that $e(Ae_i, Ae_k) = \lambda e\big(e_i, e_j\big)$ for all $i < j$ and this suffices to imply $A \in \mathrm{GSp}_{2g}(\mathbb{F}_\ell)[\lambda]$. □

Lemma 5.2 *The set of matrices in* $\mathrm{GSp}_{2g}(\mathbb{F}_\ell)[\lambda]$ *that do not have the eigenvalue* 1 *has cardinality greater than* $\beta(\ell, g)\big|\mathrm{Sp}_{2g-2}(\mathbb{F}_\ell)\big|$, *where*

$$\beta(\ell, g) = \ell^{2g-1}\big(\ell^{2g} - 1\big)\frac{\ell - 2}{\ell - 1}.$$

P r o o f. The set of matrices $A \in \mathrm{GSp}_{2g}(\mathbb{F}_\ell)[\lambda]$ that fix the vector e_1 consists of matrices of the form (5.1), where B belongs to $\mathrm{GSp}_{2g-2}(\mathbb{F}_\ell)[\lambda]$, $d, d_1, \ldots, d_{2g-2} \in \mathbb{F}_\ell$ and $b_1, \ldots, b_{2g-2}$ are given by the formula (5.2) of Lemma 5.1. Therefore the cardinality of the set of such matrices is exactly

$$\ell^{2g-1}\big|\mathrm{GSp}_{2g-2}(\mathbb{F}_\ell)[\lambda]\big| = \ell^{2g-1}\big|\mathrm{Sp}_{2g-2}(\mathbb{F}_\ell)\big|.$$

On the other hand, the symplectic group acts transitively on the set of cyclic subgroups of V (cf. [11, p. 221, Satz 9.15(a)]). Therefore if a matrix fixes any nonzero vector, it can be conjugated to one of the above. Hence, to obtain an upper bound for the number of matrices with eigenvalue 1 one has to multiply the previous number by the number of cyclic groups of V, namely $\frac{\ell^{2g}-1}{\ell-1}$. Therefore the set of matrices in $\mathrm{GSp}_{2g}(\mathbb{F}_\ell)[\lambda]$ that have the eigenvalue 1 has cardinality less than $\ell^{2g-1}\frac{\ell^{2g}-1}{\ell-1}\big|\mathrm{Sp}_{2g-2}(\mathbb{F}_\ell)\big|$. Hence the number of matrices in $\mathrm{GSp}_{2g}(\mathbb{F}_\ell)[\lambda]$ that do not have the eigenvalue 1 is greater than $\big|\mathrm{Sp}_{2g}(\mathbb{F}_\ell)\big| - \ell^{2g-1}\frac{\ell^{2g}-1}{\ell-1}\big|\mathrm{Sp}_{2g-2}(\mathbb{F}_\ell)\big|$.

Now apply the well known identity (see for instance the proof of [11, p. 220, Satz 13(b)])

$$|\mathrm{Sp}_{2g}(\mathbb{F}_\ell)| = \big(\ell^{2g} - 1\big)\ell^{2g-1}\big|\mathrm{Sp}_{2g-2}(\mathbb{F}_\ell)\big|. \tag{5.3}$$

We thus see that the set of matrices in $\mathrm{GSp}_{2g}(\mathbb{F}_\ell)[\lambda]$ that do not have the eigenvalue 1 has cardinality greater than $\beta(\ell, g)\big|\mathrm{Sp}_{2g-2}(\mathbb{F}_\ell)\big|$. □

For $\alpha = (\alpha_3, \ldots, \alpha_{2g}) \in \mathbb{F}_\ell^{2g-2}$ we put $u_\alpha := e_2 + \alpha_3 e_3 + \cdots + \alpha_{2g} e_{2g}$.

Definition 5.3 For each $\lambda \in \mathbb{F}_\ell^\times$ choose once and for all a subset $\mathcal{B}_\lambda$ of matrices $B \in \mathrm{GSp}_{2g-2}(\mathbb{F}_\ell)[\lambda]$ which do not have the eigenvalue 1, with

$$|\mathcal{B}_\lambda| = \beta(\ell, g-1)|\mathrm{Sp}_{2g-4}(\mathbb{F}_\ell)|$$

(which can be done by Lemma 5.2). Define

$$\begin{aligned} S_\lambda(\ell)_0 := \big\{ & A \text{ of the shape (5.1) in Lemma 5.1 such that:} \\ & B \in \mathcal{B}_\lambda, \\ & d_1, \ldots, d_{2g-2} \in \mathbb{F}_\ell, \\ & d \in \mathbb{F}_\ell \smallsetminus \big\{ -(b_1, \ldots, b_{2g-2})(\mathrm{Id} - B)^{-1} \big(d_1, \ \ldots, \ d_{2g-2}\big)^t \big\} \\ & \text{and such that (2) is satisfied} \big\}, \\ S_\lambda(\ell) := \big\{ & T_{u_\alpha}[\beta]^{-1} \cdot A \cdot T_{u_\alpha}[\beta] : \alpha_3, \ldots, \alpha_{2g}, \beta \in \mathbb{F}_\ell, A \in S_\lambda(\ell)_0 \big\}. \end{aligned}$$

Let q be a power of a prime $p \neq \ell$. Define

$$S^{(q)}(\ell) := \bigcup_{i=1}^{\mathrm{ord}_\ell q} S_{q^i}(\ell).$$

Define also

$$S^{(\infty)}(\ell) = \bigcup_{\lambda \in \mathbb{F}_\ell^\times} S_\lambda(\ell).$$

Remark 5.4 The sets $S^{(q)}(\ell)$ and $S^{(\infty)}(\ell)$ are not empty. Note moreover that each of the matrices in $S^{(q)}(\ell)$ and $S^{(\infty)}(\ell)$ fixes an element of V.

Proof. Let $\lambda \in \mathbb{F}_\ell^\times$. The set $S_\lambda(\ell)_0$ is non-empty, because $\mathcal{B}_\lambda \neq \emptyset$, and every $A \in S_\lambda(\ell)_0$ satisfies $Ae_1 = e_1$. Furthermore $S_\lambda(\ell)_0 \subset S_\lambda(\ell)$ as $T_v[0] = \mathrm{Id}$ for all $v \in V$. In particular $S_\lambda(\ell)$ is non-empty. Each matrix in $S_\lambda(\ell)$ is conjugate to a matrix in $S_\lambda(\ell)_0$ and hence fixes an element of V. The assertion follows from that. □

6 Special sets of symplectic matrices over $\mathbb{Z}/n\mathbb{Z}$

This section is devoted to the proof of the following result.

Theorem 6.1 *The following properties hold*:

(1) *Let q be a power of a prime number or $q = \infty$. Then*

$$\sum_\ell \frac{|S^{(q)}(\ell)|}{|\mathrm{GSp}_{2g}^{(q)}(\mathbb{F}_\ell)|} = \infty.$$

In the first case ℓ runs through all prime numbers coprime to q and in the second case through all prime numbers.

(2) *Let q be a power of a prime number p or $q = \infty$. Let $\ell_1, \ldots, \ell_r$ be distinct prime numbers. If $q \neq \infty$ assume that the ℓ_i's are different from p. Let $n = \ell_1 \ldots \ell_r$. Then*

$$\frac{|S^{(q)}(n)|}{|\mathrm{GSp}_{2g}^{(q)}(\mathbb{Z}/n\mathbb{Z})|} = \prod_{j=1}^r \frac{|S^{(q)}(\ell_j)|}{|\mathrm{GSp}_{2g}^{(q)}(\mathbb{F}_{\ell_j})|}$$

where $S^{(q)}(n) \subset \mathrm{GSp}_{2g}(\mathbb{Z}/n\mathbb{Z})$ is the set of matrices that belong to $S^{(q)}(\ell_j)$ modulo ℓ_j, for all $j = 1, \ldots, r$.

First we will prove part (1) of Theorem 6.1. We need a series of lemmata.

We can compute the cardinality of $S_\lambda(\ell)_0$ explicitly.

Lemma 6.2 *It holds that*

$$|S_\lambda(\ell)_0| = \ell^{2g-2}(\ell-1)\beta(\ell, g-1)\big|\mathrm{Sp}_{2g-4}(\mathbb{F}_\ell)\big|.$$

P r o o f. In the definition of the set $S_\lambda(\ell)_0$ there are $|\mathcal{B}_\lambda| = \beta(\ell, g-1)\big|\mathrm{Sp}_{2g-4}(\mathbb{F}_\ell)\big|$ possible choices of B, ℓ^{2g-2} possible choices of $d_1, \ldots, d_{2g-2} \in \mathbb{F}_\ell$ and $\ell - 1$ possible choices of d. □

Lemma 6.3 *Let $A \in S_\lambda(\ell)_0$ and $x \in V$. Then $Ax = x$ if and only if $x \in \mathbb{F}_\ell e_1$.*

P r o o f. We have $Ae_1 = e_1$ because the first column of A is e_1. Suppose $Ax = x$. It suffices to show that $x \in \mathbb{F}_\ell e_1$. Consider the system of equations $A\big(x_1, \ldots, x_{2g}\big)^t = \big(x_1, \ldots, x_{2g}\big)^t$ over $\mathbb{F}_\ell$. Assume first that we have a solution with $x_2 = 0$. Then the last $2g - 2$ equations boil down to

$$B\big(x_3, \ldots, x_{2g}\big)^t = \big(x_3, \ldots, x_{2g}\big)^t.$$

But since B does not have the eigenvalue 1, it follows that $x_3 = \cdots = x_{2g} = 0$, hence x belongs to the subspace $\mathbb{F}_\ell e_1$ of V generated by e_1.

Assume now that we have a solution $\big(x_1, \ldots, x_g\big)^t$ with $x_2 \neq 0$. Since 1 is not an eigenvalue of B, the matrix $\mathrm{Id} - B$ is invertible, and we can write the last $2g - 2$ equations as

$$\big(x_3/x_2, \ldots, x_{2g}/x_2\big)^t = (\mathrm{Id} - B)^{-1}\big(d_1, \ldots, d_{2g-2}\big)^t.$$

On the other hand, the first equation reads

$$d = -\big(b_1, \ldots, b_{2g-2}\big)\big(x_3/x_2, \ldots, x_{2g}/x_2\big)^t.$$

Hence

$$d = -\big(b_1, \ldots, b_{2g-2}\big)(\mathrm{Id} - B)^{-1}\big(d_1, \ldots, d_{2g-2}\big)^t.$$

But we have precisely asked that d does not satisfy such an equation, cf. Definition 5.3. □

Lemma 6.4 *Let $\alpha_3, \ldots, \alpha_{2g}, \tilde{\alpha_3}, \ldots, \tilde{\alpha}_{2g} \in \mathbb{F}_\ell$ and $\beta, \tilde{\beta} \in \mathbb{F}_\ell$. Assume that $T_{u_\alpha}[\beta]T_{u_{\tilde{\alpha}}}\big[\tilde{\beta}\big]^{-1}(e_1) = \lambda e_1$ for some $\lambda \in \mathbb{F}_\ell^\times$. Then $T_{u_\alpha}[\beta] = T_{u_{\tilde{\alpha}}}\big[\tilde{\beta}\big]$ and $\beta = \tilde{\beta}$. Furthermore, if $\beta \neq 0$, then $u_\alpha = u_{\tilde{\alpha}}$.*

P r o o f. We have

$$T_{u_\alpha}[\beta](v) = v + \beta e\big(v, e_2 + \alpha_3 e_3 + \cdots + \alpha_{2g} e_{2g}\big)\big(e_2 + \alpha_3 e_3 + \cdots + \alpha_{2g} e_{2g}\big)$$

In particular,

$$\begin{aligned}
T_{u_\alpha}[\beta](e_1) &= e_1 + \beta e_2 + \beta\alpha_3 e_3 + \cdots + \beta\alpha_{2g} e_{2g},\\
T_{u_\alpha}[\beta](e_2) &= e_2,\\
T_{u_\alpha}[\beta](e_k) &= e_k + \beta\alpha_{k+1} e_2 + \beta\alpha_{k+1}\sum_{j=3}^{2g} \alpha_j e_j \quad \text{for} \quad k \geq 3, \quad k \text{ odd},\\
T_{u_\alpha}[\beta](e_k) &= e_k - \beta\alpha_{k-1} e_2 - \beta\alpha_{k-1}\sum_{j=3}^{2g} \alpha_j e_j \quad \text{for} \quad k \geq 3, \quad k \text{ even}.
\end{aligned}$$

Hence

$$
\begin{aligned}
T_{u_\alpha}[\beta]T_{u_{\tilde\alpha}}\big[\tilde\beta\big]^{-1}(e_1) &= T_{u_\alpha}[\beta]\left(e_1-\tilde\beta e_2-\widetilde\beta\sum_{k=3}^{2g}\widetilde\alpha_k e_k\right)\\
&= e_1+\beta e_2+\beta\alpha_3 e_3+\cdots+\beta\alpha_{2g}e_{2g}\\
&\quad-\widetilde\beta e_2\\
&\quad-\widetilde\beta\sum_{\substack{k=3\\ k\ \text{odd}}}^{2g}\widetilde\alpha_k\left(e_k+\beta\alpha_{k+1}e_2+\beta\alpha_{k+1}\sum_{j=3}^{2g}\alpha_j e_j\right)\\
&\quad-\widetilde\beta\sum_{\substack{k=3\\ k\ \text{even}}}^{2g}\widetilde\alpha_k\left(e_k-\beta\alpha_{k-1}e_2-\beta\alpha_{k-1}\sum_{j=3}^{2g}\alpha_j e_j\right)\\
&= e_1+\beta e_2+\beta\alpha_3 e_3+\cdots+\beta\alpha_{2g}e_{2g}\\
&\quad-\widetilde\beta e_2\\
&\quad-\widetilde\beta\sum_{j=3}^{2g}\widetilde\alpha_j e_j\\
&\quad-\widetilde\beta\beta\left(\sum_{\substack{k=3\\ k\ \text{odd}}}^{2g}\alpha_{k+1}\widetilde\alpha_k-\sum_{\substack{k=3\\ k\ \text{even}}}^{2g}\alpha_{k-1}\widetilde\alpha_k\right)e_2\\
&\quad-\widetilde\beta\beta\sum_{j=3}^{2g}\alpha_j\left(\sum_{\substack{k=3\\ k\ \text{odd}}}^{2g}\alpha_{k+1}\widetilde\alpha_k-\sum_{\substack{k=3\\ k\ \text{even}}}^{2g}\alpha_{k-1}\widetilde\alpha_k\right)e_j.
\end{aligned}
$$

Therefore, if $T_{u_\alpha}[\beta]T_{u_{\tilde\alpha}}\big[\tilde\beta\big]^{-1}(e_1)$ is a multiple of e_1, it is necessarily equal to e_1 and moreover we have that the coefficients of the other e_k vanish, so we get the following system of equations: the equation corresponding to e_2

$$\beta-\widetilde\beta-\widetilde\beta\beta\left(\sum_{\substack{k=3\\ k\ \text{odd}}}^{2g}\alpha_{k+1}\widetilde\alpha_k-\sum_{\substack{k=3\\ k\ \text{even}}}^{2g}\alpha_{k-1}\widetilde\alpha_k\right)=0, \tag{6.1}$$

and, for each $j=3,\dots,2g$, the equation corresponding to e_j

$$\beta\alpha_j-\widetilde\beta\widetilde\alpha_j-\widetilde\beta\beta\alpha_j\left(\sum_{\substack{k=3\\ k\ \text{odd}}}^{2g}\alpha_{k+1}\widetilde\alpha_k-\sum_{\substack{k=3\\ k\ \text{even}}}^{2g}\alpha_{k-1}\widetilde\alpha_k\right)=0. \tag{6.2}$$

If $\beta=0$, then $T_{u_\alpha}[\beta]=\mathrm{Id}$ and $T_{u_\alpha}[\beta]T_{u_{\tilde\alpha}}\big[\tilde\beta\big]^{-1}(e_1)=T_{u_{\tilde\alpha}}\big[-\tilde\beta\big](e_1)=e_1-\tilde\beta e_2-\tilde\beta\tilde\alpha_3e_3-\cdots-\tilde\beta\tilde\alpha_{2g}e_{2g}$, and since this must be equal to λe_1, we conclude that $\tilde\beta=0$, and $T_{u_{\tilde\alpha}}\big[\tilde\beta\big]=\mathrm{Id}$. Similarly if $\tilde\beta=0$, then $\beta=0$ and $T_{u_{\tilde\alpha}}\big[\tilde\beta\big]=\mathrm{Id}=T_{u_\alpha}[\beta]$.

Assume now that $\beta\neq0$, $\tilde\beta\neq0$. From Equation (6.1) we obtain that

$$\left(\sum_{\substack{k=3\\ k\ \text{odd}}}^{2g}\alpha_{k+1}\widetilde\alpha_k-\sum_{\substack{k=3\\ k\ \text{even}}}^{2g}\alpha_{k-1}\widetilde\alpha_k\right)=\frac{\beta-\tilde\beta}{\beta\tilde\beta};$$

substituting this in Equation (6.2) we get that $\alpha_j=\widetilde\alpha_j$, and once we have this for all $j=3,\dots,2g$, it follows from Equation (6.1) that $\beta=\widetilde\beta$. □

Lemma 6.5 *Let $A, \tilde{A} \in S_\ell(\lambda)_0$. Assume that there exist $\alpha_3, \ldots, \alpha_{2g}, \tilde{\alpha_3}, \ldots, \tilde{\alpha}_{2g} \in \mathbb{F}_\ell$ and $\beta, \tilde{\beta} \in \mathbb{F}_\ell$ such that $T_{u_\alpha}[\beta]^{-1} \cdot A \cdot T_{u_\alpha}[\beta] = T_{u_{\tilde{\alpha}}}\big[\widetilde{\beta}\big]^{-1} \cdot \widetilde{A} \cdot T_{u_{\tilde{\alpha}}}\big[\widetilde{\beta}\big]$. Then $A = \widetilde{A}$ and $\beta = \tilde{\beta}$. If $\beta \neq 0$, then $\alpha_i = \tilde{\alpha}_i$ for all $i \geq 3$.*

P r o o f. Since $\widetilde{A} = T_{u_{\tilde{\alpha}}}\big[\widetilde{\beta}\big] T_{u_\alpha}[\beta]^{-1} \cdot A \cdot T_{u_\alpha}[\beta] T_{u_{\tilde{\alpha}}}\big[\widetilde{\beta}\big]^{-1}$ fixes e_1, we see that A fixes $T_{u_\alpha}[\beta] T_{u_{\tilde{\alpha}}}\big[\widetilde{\beta}\big]^{-1} e_1$. Lemma 6.3 implies $T_{u_\alpha}[\beta] T_{u_{\tilde{\alpha}}}\big[\widetilde{\beta}\big]^{-1} e_1 = \lambda e_1$ for some $\lambda \in \mathbb{F}_\ell^\times$. The assertion follows from that by Lemma 6.4. □

Next we can compute the cardinality of $S_\lambda(\ell)$ in terms of $|S_\lambda(\ell)_0|$.

Lemma 6.6 $|S_\lambda(\ell)| = \big(\ell^{2g-2}(\ell-1)+1\big)|S_\lambda(\ell)_0|$.

P r o o f. For $A \in S_\lambda(\ell)_0$ we define

$$C_A = \big\{T_{u_\alpha}[\beta] A T_{u_\alpha}[\beta]^{-1} : \alpha_3, \ldots, \alpha_{2g}, \beta \in \mathbb{F}_\ell\big\}.$$

Lemma 6.5 implies that $|C_A| = \ell^{2g-2}(\ell-1)+1$ and that $C_A \cap C_{A'} = \emptyset$ for $A \neq A'$ in $S_\lambda(\ell)_0$. Furthermore $S_\lambda(\ell) = \bigcup_{A \in S_\lambda(\ell)_0} C_A$, cf. Definition 5.3. Thus $|S_\lambda(\ell)| = \big(\ell^{2g-2}(\ell-1)+1\big)|S_\lambda(\ell)_0|$. □

Lemma 6.7

(1) *Let q be a power of a prime number p, and let n be a squarefree natural number such that $p \nmid n$. The cardinality of $\mathrm{GSp}_{2g}^{(q)}(\mathbb{Z}/n\mathbb{Z})$ equals $\mathrm{ord}_n(q) \cdot \prod_{\ell|n} |\mathrm{Sp}_{2g}(\mathbb{F}_\ell)|$.*

(2) *Let $q = \infty$, and let n be a squarefree natural number. The cardinality of $\mathrm{GSp}_{2g}^{(q)}(\mathbb{Z}/n\mathbb{Z})$ equals $\prod_{\ell|n}(\ell-1)|\mathrm{Sp}_{2g}(\mathbb{F}_\ell)|$.*

P r o o f. By the Chinese Remainder Theorem $|\mathrm{Sp}(\mathbb{Z}/n\mathbb{Z})| = \prod_{\ell|n} |\mathrm{Sp}(\mathbb{Z}/\ell\mathbb{Z})|$. Furthermore the multiplicator map $\varepsilon\colon \mathrm{GSp}_{2g}(\mathbb{Z}/n\mathbb{Z}) \to (\mathbb{Z}/n\mathbb{Z})^\times$ is an epimorphism with kernel $\mathrm{Sp}_{2g}(\mathbb{Z}/n\mathbb{Z})$. Thus

$$\big|\mathrm{GSp}_{2g}^{(\infty)}(\mathbb{Z}/n\mathbb{Z})\big| = |\mathrm{GSp}_{2g}(\mathbb{Z}/n\mathbb{Z})| = |\mathrm{Sp}(\mathbb{Z}/n\mathbb{Z})|\big|(\mathbb{Z}/n\mathbb{Z})^\times\big|.$$

Furthermore $|(\mathbb{Z}/n\mathbb{Z})^\times| = \prod_{\ell|n}(\ell-1)$. Hence (2) holds true.

Now let q be a prime power which is coprime to n. It follows from the definitions that $\mathrm{GSp}_{2g}^{(q)}(\mathbb{Z}/n\mathbb{Z}) = \varepsilon^{-1}(\langle\overline{q}\rangle)$. Thus

$$\big|\mathrm{GSp}_{2g}^{(q)}(\mathbb{Z}/n\mathbb{Z})\big| = |\langle\overline{q}\rangle||\ker(\varepsilon)| = |\langle\overline{q}\rangle||\mathrm{Sp}_{2g}(\mathbb{Z}/n\mathbb{Z})|.$$

This implies (1) because $\mathrm{ord}_n(q) = |\langle\overline{q}\rangle|$. □

P r o o f o f T h e o r e m 6.1(1). Let q be a power of a prime p or $q = \infty$, and let ℓ be a prime. In the first case, let us also assume $\ell \neq p$. In the first case define $G = \langle\overline{q}\rangle \subset \mathbb{F}_\ell^\times$; then $|G| = \mathrm{ord}_\ell(q)$. In the second case define $|G| = \mathbb{F}_\ell^\times$; then $|G| = (\ell-1)$. In both cases $\big|\mathrm{GSp}_{2g}^{(q)}(\mathbb{F}_\ell)\big| = |G|\big|\mathrm{Sp}_{2g}(\mathbb{F}_\ell)\big|$ by Lemma 6.7. Furthermore

$$\big|\mathrm{Sp}_{2g}(\mathbb{F}_\ell)\big| = \big(\ell^{2g}-1\big)\ell^{2g-1}\big|\mathrm{Sp}_{2g-2}(\mathbb{F}_\ell)\big| = \big(\ell^{2g}-1\big)\ell^{2g-1}\big(\ell^{2g-2}-1\big)\ell^{2g-3}\big|\mathrm{Sp}_{2g-4}(\mathbb{F}_\ell)\big|.$$

(cf. the identity (3) in the proof of Lemma 5.2). Thus

$$\big|\mathrm{GSp}_{2g}^{(q)}(\mathbb{F}_\ell)\big| = |G|\big(\ell^{2g}-1\big)\ell^{2g-1}\big(\ell^{2g-2}-1\big)\ell^{2g-3}\big|\mathrm{Sp}_{2g-4}(\mathbb{F}_\ell)\big| \quad (1).$$

Furthermore $S^{(q)}(\ell) = \bigcup_{\lambda \in G} S_\lambda(\ell)$ and hence

$$\big|S^{(q)}(\ell)\big| = |G||S_\lambda(\ell)| = |G|\big(\ell^{2g-2}(\ell-1)+1\big)|S_\lambda(\ell)_0|$$

by Lemma 6.6. Recall from Lemma 6.2 that

$$|S_\lambda(\ell)_0| = \ell^{2g-2}(\ell-1)\beta(\ell, g-1)\big|\mathrm{Sp}_{2g-4}(\mathbb{F}_\ell)\big|.$$

It follows that

$$\left|S^{(q)}(\ell)\right| = |G|\left(\ell^{2g-2}(\ell-1)+1\right)\ell^{2g-2}(\ell-1)\beta(\ell,g-1)\left|\mathrm{Sp}_{2g-4}(\mathbb{F}_\ell)\right| \quad (2)$$

Dividing Equation (1) by Equation (2) we obtain

$$\frac{\left|S^{(q)}(\ell)\right|}{\left|\mathrm{GSp}_{2g}^{(q)}(\mathbb{F}_\ell)\right|} = \frac{\left(\ell^{2g-2}(\ell-1)+1\right)\ell^{2g-2}(\ell-1)\beta(\ell,g-1)}{\left(\ell^{2g}-1\right)\ell^{2g-1}\left(\ell^{2g-2}-1\right)\ell^{2g-3}} \sim \frac{1}{\ell},$$

and the sum $\sum_{\ell\neq p \text{ prime}} \frac{1}{\ell}$ diverges. □

For the rest of the section, q will be a power of a prime p.

For each squarefree n not divisible by p and each $i = 1, \ldots, \mathrm{ord}_n(q)$, define the set $S_{q^i}(n) := \left\{A \in S^{(q)}(n) \mid \varepsilon(A) = q^i \text{ modulo } n\right\}$.

Lemma 6.8 *Let q be a power of a prime number p. Let $\ell_1, \ldots, \ell_r$ be distinct primes which are different from p, and consider $n = \ell_1 \cdot \cdots \cdot \ell_r$. Let $i \in \{1, \ldots, \mathrm{ord}_n(q)\}$. Then there is a bijection*

$$S_{q^i}(n) \simeq S_{q^i}(\ell_1) \times \cdots \times S_{q^i}(\ell_r).$$

Proof. Consider the canonical projection

$$\begin{aligned}\pi : S_{q^i}(n) &\longrightarrow S_{q^i}(\ell_1) \times \cdots \times S_{q^i}(\ell_r)\\ A &\longmapsto (A \mod \ell_1, \ldots, A \mod \ell_r).\end{aligned}$$

This is clearly an injective map. Now we want to prove surjectivity. For each j, take some matrix $B_j \in S_{q^i}\left(\ell_j\right)$.

By the Chinese Remainder Theorem, there exists $A \in \mathrm{GSp}_{2g}(\mathbb{Z}/n\mathbb{Z})$ such that A projects onto B_j for each j. Note that in particular $A \in S^{(q)}(n)$. Since $\varepsilon(A)$ is congruent to $\varepsilon(B_j) = q^i$ modulo ℓ_j for all j, we get that $\varepsilon(A) = q^i$ modulo n. Therefore $A \in S_{q^i}(n)$. □

Proof of Theorem 6.1(2).

Case $q \neq \infty$: On the one hand, since the cardinality of $|S_{q^i}(\ell)|$ does not depend on i (cf. Lemmas 6.6 and 6.2), we obtain

$$\prod_{\ell|n} \frac{\left|S^{(q)}(\ell)\right|}{\left|\mathrm{GSp}_{2g}^{(q)}(\mathbb{F}_\ell)\right|} = \prod_{\ell|n} \frac{\mathrm{ord}_\ell(q)|S_q(\ell)|}{\mathrm{ord}_\ell(q)\left|\mathrm{Sp}_{2g}(\mathbb{F}_\ell)\right|} = \prod_{\ell|n} \frac{|S_q(\ell)|}{\left|\mathrm{Sp}_{2g}(\mathbb{F}_\ell)\right|} \quad (1).$$

On the other hand, taking into account again that $\left|S_{q^i}(\ell)\right|$ is independent of i, Lemma 6.7, and that $\left|S_{q^i}(n)\right| = \prod_{\ell|n}\left|S_{q^i}(\ell)\right|$ by Lemma 6.8, we get

$$\begin{aligned}\frac{\left|S^{(q)}(n)\right|}{\left|\mathrm{GSp}_{2g}^{(q)}(\mathbb{Z}/n\mathbb{Z})\right|} &= \frac{\sum_{i=1}^{\mathrm{ord}_n(q)}\left|S_{q^i}(n)\right|}{\mathrm{ord}_n(q)\prod_{\ell|n}\left|\mathrm{Sp}_{2g}(\mathbb{F}_\ell)\right|} = \frac{\sum_{i=1}^{\mathrm{ord}_n(q)}\left(\prod_{\ell|n}\left|S_{q^i}(\ell)\right|\right)}{\mathrm{ord}_n(q)\prod_{\ell|n}\left|\mathrm{Sp}_{2g}(\mathbb{F}_\ell)\right|}\\ &= \frac{\sum_{i=1}^{\mathrm{ord}_n(q)}\left(\prod_{\ell|n}\left|S_q(\ell)\right|\right)}{\mathrm{ord}_n(q)\prod_{\ell|n}\left|\mathrm{Sp}_{2g}(\mathbb{F}_\ell)\right|} = \frac{\mathrm{ord}_n(q)\left(\prod_{\ell|n}\left|S_q(\ell)\right|\right)}{\mathrm{ord}_n(q)\prod_{\ell|n}\left|\mathrm{Sp}_{2g}(\mathbb{F}_\ell)\right|}\\ &= \prod_{\ell|n}\frac{\left|S_q(\ell)\right|}{\left|\mathrm{Sp}_{2g}(\mathbb{F}_\ell)\right|} \quad (2).\end{aligned}$$

Our assertion follows comparing the right-hand sides of (1) and (2).

Case $q = \infty$: By the Chinese Remainder Theorem, there is a canonical isomorphism

$$c\colon \mathrm{GSp}_{2g}^{(\infty)}(\mathbb{Z}/n\mathbb{Z}) \cong \prod_{i=1}^{r} \mathrm{GSp}_{2g}^{(\infty)}(\mathbb{Z}/\ell_i\mathbb{Z})$$

and

$$S^{(\infty)}(n) = c^{-1}\big(S^{(\infty)}(\ell_1) \times \cdots \times S^{(\infty)}(\ell_r)\big)$$

by the definition of $S^{(\infty)}(n)$. It follows that

$$\frac{\big|S^{(\infty)}(n)\big|}{\big|\mathrm{GSp}_{2g}^{(\infty)}(\mathbb{Z}/n\mathbb{Z})\big|} = \prod_{i=1}^{r} \frac{\big|S^{(\infty)}(\ell_i)\big|}{\big|\mathrm{GSp}_{2g}^{(\infty)}(\mathbb{Z}/\ell_i\mathbb{Z})\big|} \qquad \square$$

Remark 6.9 In the definition of the set $S_{q^i}(\ell)_0$ (cf. Definition 5.3), we choose a subset $\mathcal{B}_{q^i}$ of matrices in $\mathrm{GSp}_{2g-2}(\mathbb{F}_\ell)[q^i]$ without the eigenvalue 1, which is large enough to ensure that part (1) of Theorem 6.1 holds. For a concrete value of g, one can choose such set more explicitly. For instance, when $g = 2$, instead of $\mathcal{B}_{q^i}$ one can consider the set

$$\mathcal{B}'_{q^i} := \left\{ \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} : b_{1,1} \in \mathbb{F}_\ell,\ b_{2,2} \in \mathbb{F}_\ell \smallsetminus \big\{1 - b_{1,1} + q^i\big\}, b_{1,2} \in \mathbb{F}_\ell^\times,\ b_{2,1} = b_{1,2}^{-1}\big(b_{1,1}b_{2,2} - q^i\big) \right\}$$

of $\ell(\ell-1)^2$ matrices, which can also be used to prove the second part of Theorem 6.1 in the case of the group $\mathrm{GSp}_4(\mathbb{F}_\ell)$.

7 Proof of the Conjecture of Geyer and Jarden, part (a)

Theorem 7.1 *Let (A, λ) be a polarized abelian variety over a finitely generated field K. Assume that A/K has big monodromy. Then for almost all $\sigma \in G_K$ there are infinitely many prime numbers ℓ such that $A(K_{\mathrm{sep}}(\sigma))[\ell] \neq 0$.*

Proof. Let $p := \mathrm{char}(K)$. Let $G = G_K$ and $g := \dim(A)$. Denote by $e_{\ell^\infty}^{\mathrm{can}}$ (resp. e_n^{can}) the canonical symplectic pairing on $T_\ell(A)$ (resp. $A[n]$), cf. the beginning of Section 5. Recall from Section 2 that we have furthermore the Weil pairing $e_{\ell^\infty}^{\lambda}$ (resp. e_n^λ) on $T_\ell(A)$ (resp. on $A[n]$). We fix once and for all for every prime number $\ell \neq p$, $\ell > \deg(\lambda)$ a symplectic basis of $\big(T_\ell(A), e_{\ell^\infty}^\lambda\big)$ (cf. [2, Chap. 9, paragraph 5, no. 1, Thm. 1, p. 79]). This defines an isometry $\big(T_\ell(A), e_{\ell^\infty}^\lambda\big) \cong (\mathbb{Z}_\ell, e_{\ell^\infty}^{\mathrm{can}})$, from which we obtain an isometry $\big(A[\ell^i], e_{\ell^i}^\lambda\big) \cong \big(A[\ell^i], e_{\ell^i}^{\mathrm{can}}\big)$ for every i. Finally, by the Chinese remainder theorem, we obtain an isometry $\big(A[n], e_n^\lambda\big) \cong \big((\mathbb{Z}/n\mathbb{Z})^{2g}, e_n^{\mathrm{can}}\big)$ for every n which is coprime to p and to $\deg(\lambda)$. We get an isomorphism $\mathrm{GSp}\big(A[n], e_n^\lambda\big) \cong \mathrm{GSp}_{2g}(\mathbb{Z}/n\mathbb{Z})$ for every such n, and we consider the representations

$$\rho_n : G_K \longrightarrow \mathrm{GSp}_{2g}(\mathbb{Z}/n\mathbb{Z})$$

attached to A/K after these choices. If m is a divisor of n, then we denote by $r_{n,m} : \mathrm{GSp}_{2g}(\mathbb{Z}/n\mathbb{Z}) \to \mathrm{GSp}_{2g}(\mathbb{Z}/m\mathbb{Z})$ the corresponding canonical map, such that $r_{n,m} \circ \rho_n = \rho_m$.

Let $q := q(K)$ be the cardinality of the algebraic closure of the prime field of K in K. Thus $q = \infty$ if $p = 0$ and q is a power of p otherwise. As A has big monodromy, we find by Proposition 3.4 an integer c (divisible by $\deg(\lambda)$ and by p, if $p \neq 0$) such that $\mathrm{im}(\rho_n) = \mathrm{GSp}_{2g}^{(q)}(\mathbb{Z}/n\mathbb{Z})$, for every n coprime to c.

For every prime number $\ell > c$, we define

$$X_\ell := \{\sigma \in G_K \mid A(K_{\mathrm{sep}}(\sigma))[\ell] \neq 0\}.$$

Thus, it suffices to prove that $\bigcap_{n>c} \bigcup_{\ell \geq n \text{ prime}} X_\ell$ has measure 1. Let $S^{(q)}(n) \subset \mathrm{GSp}_{2g}^{(q)}(\mathbb{Z}/n\mathbb{Z})$ be the special sets of symplectic matrices defined in Section 4. By Remark 5.4 $\rho_\ell^{-1}\big(S^{(q)}(\ell)\big) \subset X_\ell$ for every prime number $\ell > c$. Thus is suffices to prove that $\bigcap_{n>c} \bigcup_{\ell \geq n \text{ prime}} \rho_\ell^{-1}\big(S^{(q)}(\ell)\big)$ has measure 1. By the basic properties of the Haar measure, $\mu_G\big(\rho_n^{-1}\big(S^{(q)}(n)\big)\big) = \frac{|S^{(q)}(n)|}{|\mathrm{GSp}_{2g}^{(q)}(\mathbb{Z}/n\mathbb{Z})|}$ for all integers n coprime to c. Hence part (1) of Theorem 6.1 implies that $\sum_{\ell > c \text{ prime}} \mu_G\big(\rho_\ell^{-1}\big(S^{(q)}(\ell)\big)\big) = \infty$.

Furthermore, if $\ell_1, \dots, \ell_r > c$ are distinct prime numbers and $n = \ell_1 \dots \ell_r$, then

$$\bigcap_{i=1}^{r} \rho_{\ell_i}^{-1}\big(S^{(q)}(\ell_i)\big) = \rho_n^{-1}\big(S^{(q)}(n)\big)$$

and part (2) of Theorem 6.1 implies

$$\mu_G\left(\bigcap_{i=1}^{r} \rho_{\ell_i}^{-1}\big(S^{(q)}(\ell_i)\big)\right) = \prod_{i=1}^{r} \mu_G\big(\rho_{\ell_i}^{-1}\big(S^{(q)}(\ell_i)\big)\big).$$

Hence $\big(\rho_\ell^{-1}\big(S^{(q)}(\ell)\big)\big)_{\ell > c}$ is a μ_G-independent sequence of subsets of G. It follows from Lemma 4.3 that $\bigcap_{n>c} \bigcup_{\ell \geq n \text{ prime}} \rho_\ell^{-1}\big(S^{(q)}(\ell)\big)$ has measure 1, as desired. □

We now combine the main theorems 4.1 and 7.1 of this paper with existing computations of monodromy groups. We will obtain many examples of abelian varieties for which the Conjecture of Geyer and Jarden can be shown. Certainly, the most prominent monodromy computation is the classical theorem of Serre (cf. [19], [20] for the number field case; the generalization to finitely generated fields of characteristic zero is well-known): *If A is an abelian variety over a finitely generated field K of characteristic zero with* $\mathrm{End}(A) = \mathbb{Z}$ *and* $\dim(A) = 2, 6$ *or odd, then A/K has big monodromy.* Here $\mathrm{End}(A) = \mathrm{End}_{\tilde{K}}(A_{\tilde{K}})$ stands for the absolute endomorphism ring of A.

Furthermore we focus our attention at abelian varieties with $\mathrm{End}(A) = \mathbb{Z}$, which have been recently considered by Chris Hall in his open image theorem [10]. We will say that an abelian variety A over a finitely generated field K *is of Hall type*, if $\mathrm{End}(A) = \mathbb{Z}$ and K has a discrete valuation v such that the connected component of the special fibre of the Néron model $\mathcal{A} \to \mathrm{Spec}(\mathcal{O}_v)$ of A over the discrete valuation ring $\mathcal{O}_v$ of v is an extension of an abelian variety by a 1-dimensional torus. The following result, gives examples of abelian varieties with big monodromy in all dimensions (and including the case $\mathrm{char}(K) > 0$): *If A is an abelian variety of Hall type over a finitely generated infinite field K, then A/K has big monodromy.* In the special case where K is a global field this has recently been proved by Hall (cf. [9], [10]). The generalization to an arbitrary finitely generated ground field K is carried out in our paper [1] using methods of group theory, finiteness properties of the fundamental group of schemes and Galois theory of large field extensions. In combination with the main theorem we obtain the following

Corollary 7.2 *Let A be an abelian variety over a finitely generated infinite field K. Assume that either condition* (i) *or* (ii) *is satisfied.*

(i) *A is of Hall type.*

(ii) $\mathrm{char}(K) = 0$*,* $\mathrm{End}(A) = \mathbb{Z}$ *and* $\dim(A) = 2, 6$ *or odd.*

Then the Conjecture of Geyer and Jarden holds true for A/K.

We thus obtain over every finitely generated infinite field and for every dimension families of abelian varieties for which the Conjecture of Geyer and Jarden holds true. In the case when $\mathrm{char}(K) > 0$ the corollary offers the first evidence for the Conjecture of Geyer and Jarden on torsion going beyond the case of elliptic curves.

Acknowledgements We warmly thank Gerhard Frey, Dieter Geyer, Cornelius Greither and Moshe Jarden for conversations and useful comments on the topic of this paper. S. A. is a research fellow of the Alexander von Humboldt Foundation. S. A. was partially supported by the Ministerio de Educación y Ciencia grant MTM2009-07024. S. A. wants to thank the Hausdorff Research Institute for Mathematics in Bonn, the Centre de Recerca Matemàtica in Bellaterra and the Mathematics Department of Adam Mickiewicz University in Poznań for their support and hospitality while she worked on this project. W. G. and S. P. were partially supported by the Deutsche Forschungsgemeinschaft research grant GR 998/5-1. W. G. was partially supported by the Alexander von Humboldt Research Fellowship and the research NCN grant N N201 607440 "Arithmetic Geometry". W. G. thanks the Centre Recerca Matemàtica in Bellaterra and the Max Planck Institut für Mathematik in Bonn for support and hospitality during visits in 2010, when he worked on this project. S. P. gratefully acknowledges the hospitality of Mathematics Department of Adam Mickiewicz University in Poznań and of the Minkowski Center at Tel Aviv University during several research visits.

References

[1] S. Arias-de-Reyna, W. Gajda, and S. Petersen, Big monodromy theorem for abelian varieties over finitely generated fields, J. Pure Appl. Algebra, doi:10.1016/j.jpaa.2012.06.010 (2012).
[2] N. Bourbaki, Algèbre, Éléments de Mathématique (Springer, 2007).
[3] B. Conrad, Polarizations, Manuscript.
[4] M. D. Fried and M. Jarden, Field arithmetic (2nd revised and enlarged ed. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3, Folge 11, Springer, Berlin, 2005).
[5] W. D. Geyer and M. Jarden, Torsion points of elliptic curves over large algebraic extensions of finitely generated fields, Israel J. Math. **31**, 157–197 (1978).
[6] W. D. Geyer and M. Jarden, Torsion of Abelian varieties over large algebraic fields, Finite Fields Appl. **11**(1), 123–150 (2005).
[7] W. D. Geyer and M. Jarden, The rank of abelian varieties over large algebraic fields. Arch. Math. **86**(3), 211–216 (2006).
[8] L. Grove, Classical groups and geometrical algebra (Graduate Studies in Mathematics 30, American Mathematical Society, 2002).
[9] C. Hall, Big symplectic or orthogonal monodromy modulo l, Duke Math. J. **141**(1), 179–203 (2008).
[10] C. Hall, An open-image theorem for a general class of abelian varieties, Bull. Lond. Math. Soc. **43**(4), 703–711 (2011).
[11] B. Huppert, Endliche Gruppen (Springer-Verlag, 1976).
[12] B. H. Im and M. Larsen, Abelian varieties over cyclic fields, Amer. J. Math. **130**(5), 1195–1210 (2008).
[13] M. Jacobson and M. Jarden, Finiteness theorems for torsion of abelian varieties over large algebraic fields, Acta Arith. **98**, 15–31 (2001).
[14] M. Larsen, Maximality of Galois actions for compatible systems, Duke Math. J. **80**(3), 601–630 (1995).
[15] M. Larsen, Rank of elliptic curves over almost separably closed fields, Bull. Lond. Math. Soc. **35**(6), 817–820 (2003).
[16] R. Lidl and H. Niederreiter, Finite Fields (Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1997).
[17] J. Milne, Abelian Varieties, in: Arithmetic Geometry, edited by G. Cornell and J. Silverman (Springer, 1986).
[18] K. Ribet, Torsion points of abelian varieties in cyclotomic extensions (appendix to an article of N. Katz and S. Lang), Enseign. Math. **27**(3–4), 285–319 (1981).
[19] J. P. Serre, Résumé des cours de 1984-1985 (Annuaire du Collège de France, 1985).
[20] J. P. Serre, Résumé des cours de 1985-1986 (Annuaire du Collège de France, 1986).
[21] J. P. Serre, Lettre á Marie-France Vignéras du 10/2/1986, in: Collected Papers IV (Springer-Verlag, 2000).
[22] D. Taylor, The Geometry of the Classical Groups (Heldermann Verlag, 1992).
[23] T. Weigel, On profinite completion of arithmetic groups of split type, in: Lois d' Algébre et Variétés Algébriques, edited by M. Gonze (Hermann, 1996).
[24] Y. Zarhin, Endomorphisms and torsion of abelian varieties, Duke Math. Jour. **54**(1), 131–145 (1987).
[25] D. Zywina, Abelian varieties over large algebraic fields with infinite torsion, preprint (2010), available at http://www.arxiv.org

Query

Q1: AU: Please provide Fax number.

Instructions for Proof Corrections and Orders

MATHEMATISCHE NACHRICHTEN

2013

Dr. Marilene Teixeira Balbi
Mathematische Nachrichten
Carl-Heindl-Strasse 4
93077 Bad Abbach
Germany

TEL +49 (0) 94 05–95 75 67
FAX +49 (0) 94 05–95 75 68
E-MAIL marilene.balbi@t-online.de

WILEY Publishers Since 1807

Please correct your proofs and return them within 14 days together with the completed reprint order form. The editors reserve the right to publish your article with editors' corrections if your proofs do not arrive in time.

After having received your corrections, your paper will be published online soon in the Wiley Online Library (wileyonlinelibrary.com).
Please keep in mind that reading proofs is your responsibility. Corrections should therefore be clear. The use of standard proof correction marks is recommended. Corrections listed in an electronic file should be sorted by line numbers. Please do not incorporate your corrections into the electronic PDF file.
Manuscript files are sometimes slightly modified by the production department to follow general presentation rules of the journal.

Note that the quality of the halftone figures is not as high as the final version that will appear in the issue.

Check the enclosed proofs very carefully, paying particular attention to the formulas (including line breakings introduced in production), figures, numerical values, tabulated data and layout of the pages.

A black box (■) or a question at the end of the paper (after the references) signals unclear or missing information that specifically requires **your attention.** Note that the author is liable for damages arising from incorrect statements, including misprints.

The main aim of proofreading is to correct errors that may have occurred during the production process, **and not to modify the content of the paper**. Corrections that may lead to a change in the page layout should be avoided.

Note that sending back a corrected **manuscript** file **is of no use.**

Return the corrected proofs within 14 days by e-mail.

Please do not send your corrections to the typesetter but to the Editorial Office:

E-MAIL marilene.balbi@t-online.de

Please limit corrections to errors in the text; cost incurred for any further changes or additions will be charged to the author, unless such changes have been agreed upon by the editor.

If your paper contains **color figures**, please fill in the Color Print Authorization and note the further information given on the following pages.

Full color reprints, Customized PDF files, Printed Issues, Color Print, and Cover Posters may be ordered by filling in the accompanying form.

Contact the Editorial Office for **special offers** such as

- Personalized and customized reprints (e.g. with special cover, selected or all your articles published in Wiley-VCH journals)
- Cover/frontispiece publications and posters (standard or customized)
- Promotional packages accompanying your publication

Visit the **MaterialsViews.com Online Store** for a wide selection of posters, logos, prints and souvenirs from our top physics and materials science journals at www.cafepress.com/materialsviews

Order Form

MATHEMATISCHE NACHRICHTEN

2013

Dr. Marilene Teixeira Balbi
Mathematische Nachrichten
Carl-Heindl-Strasse 4
93077 Bad Abbach
Germany

TEL +49 (0) 94 05–95 75 67
FAX +49 (0) 94 05–95 75 68
E-MAIL marilene.balbi@t-online.de

WILEY Publishers Since 1807

Please complete this form and return it by e-mail or FAX.

Article No. ______

Author/Title ______

e-mail address ______

Required Fields may be filled in using Adobe Reader

Color Print Authorization

Please bill me for

☐ color print figures (total number of color figures)

❐ YES, please print Figs. No. ______ in color.

❐ NO, please print all color figures in black/white.

Reprints/Issues/PDF Files/Posters

Whole issues, reprints and PDF files (300 dpi) for an unlimited number of printouts are available at the rates given on the next page. Reprints and PDF files can be ordered before *and after* publication of an article. All reprints will be delivered in full color, regardless of black/white printing in the journal.

Reprints

Please send me and bill me for

☐ full color reprints with color cover

☐ full color reprints with personalized color cover

Issues

Please send me and bill me for

☐ entire issues

Customized PDF-Reprint

Please send me and bill me for

❐ a PDF file (300 dpi) for an unlimited number of printouts **with customized color cover sheet.**

The PDF file will be sent to your e-mail address.

Send PDF file to: ______

Please note that posting of the final published version on the open internet is not permitted. For author rights and re-use options, see the Copyright Transfer Agreement at http://www.wiley.com/go/ctavchglobal.

Cover Posters

Posters are available of all the published covers in two sizes (see attached price list). **Please send me and bill me for**

☐ A2 (42 × 60 cm/17 × 24in) posters

☐ A1 (60 × 84 cm/24 × 33in) posters

Mail reprints and/or issues and/or posters to (no P.O. Boxes):

______

VAT number: ______

Information regarding VAT

Please note that from German sales tax point of view, the charge for **Reprints, Issues or Posters** is considered as **"supply of goods"** and therefore, in general, such delivery is a subject to German sales tax. However, this regulation has no impact on customers located outside of the European Union. Deliveries to customers outside the Community are automatically tax-exempt. Deliveries within the Community to institutional customers outside of Germany are exempted from the German tax (VAT) only if the customer provides the supplier with his/her VAT number.

The VAT number (value added tax identification number) is a tax registration number used in the countries of the European Union to identify corporate entities doing business there. It starts with a country code (e.g. FR for France, GB for Great Britain) and follows by numbers.

The charge for **front cover/back cover/inside cover pictures, color figures or frontispieces publications** is considered as **"supply of services"** and therefore it is a subject to German sales tax. However, in case you are an institutional customer outside of Germany, the tax can be waived if you provide us with the VAT number of your company.

Customers outside of the EU may have a VAT number starting with "EU" instead of the country code if they are registered by the EU's tax authorities. In case you do not have a VAT number of EU and you are a taxable person doing business in a country outside EU, then please provide us with a certification from your local tax authorities confirming that you are a taxable person under the local tax law. Please note that the certification needs to confirm that you are a taxable person and you are conducting an economic activity in your country. Certifications which confirm that you are tax-exempt legal body (non-profit organization, public body, school, political party, etc.) in your country cannot be accepted for the German VAT purposes.

Purchase Order No.: ______

Terms of payment:

❑ Please send an invoice ❑ Cheque is enclosed

❑ **VISA, MasterCard and AMERICAN EXPRESS.**

Please use this link (Credit Card Token Generator) to create a secure Credit Card Token and include this number in the form instead of the credit card data.

https://www.wiley-vch.de/editorial_production/index.php

CREDIT CARD TOKEN NUMBER:

Send invoice to:

______

Signature ______

Date ______

Please use this form to confirm that you are prepared to pay your contribution.

Please sign and return this page.

You will receive an invoice following the publication of your article in the journal issue.

Price List – Mathematische Nachrichten 2013

Reprints/Issues/PDF-Files/Posters

The prices listed below are valid only for orders received in the course of 2013. Minimum order for reprints is 50 copies. **Reprints can be ordered before *and after* publication of an article. All reprints are delivered with color cover and color figures**. If more than 500 copies are ordered, special prices are available upon request.
Single issues are available to authors at a reduced price.
The prices include mailing and handling charges. All prices are subject to local VAT/sales tax.

Reprints with color cover	Price for orders of (in Euro)					
Size (pages)	50 copies	100 copies	150 copies	200 copies	300 copies	500 copies*
1–4	345	395	425	445	548	752
5–8	490	573	608	636	784	1077
9–12	640	739	786	824	1016	1396
13–16	780	900	958	1004	1237	1701
17–20	930	1070	1138	1196	1489	2022
for every additional 4 pages	147	169	175	188	231	315
for personalized color cover	**190**	**340**	**440**	**650**	**840**	**990**

PDF file (300 dpi, unlimited number of printouts, customized cover sheet)	€ 330
Issues	€ 48 per copy for up to 10 copies.*
Cover Posters	• A2 (42 × 60 cm/17 × 24in) € 49 • A1 (60 × 84 cm/24 × 33in) € 69

*Prices for more copies available on request.

Special offer: If you order 100 or more reprints you will receive a pdf file (300 dpi, unlimited number of printouts, color figures) and an issue for free.

Color figures

If your paper contains **color figures**, please notice that, generally, these figures will appear in color in the online PDF version and all reprints of your article at no cost. This will be indicated by a note "(online color at: www.mn-journal.org)" in the caption. The print version of the figures in the journal hardcopy will be black/white unless the author explicitely requests a color print publication and contributes to the additional printing costs.

	Approximate color print figure charges
First figure	€ 495
Each additional figure	€ 395 Special prices for more color print figures on request

If you wish color figures in print, please answer the **color print authorization** questions on the order form.