Iwasawa Main Conjecture in cyclotomic function fields

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Cyclotomic extensions: number fields

For any $n \in \mathbb{N}$ and p an odd prime, let

- $\mathbb{Q}_n := \mathbb{Q}(\boldsymbol{\mu}_{p^{n+1}})$

$$\mathbb{Q} \overset{p-1}{\subset} \mathbb{Q}_0 \overset{p}{\subset} \mathbb{Q}_1 \overset{p}{\subset} \cdots \subset \mathbb{Q}_n \subset \cdots \subset \bigcup \mathbb{Q}(\boldsymbol{\mu}_{p^n}) = \mathbb{Q}(\boldsymbol{\mu}_{p^\infty}) \ .$$

- $\mathbb{Q}_{cyc} := \bigcup \mathbb{Q}_n$ the p-cyclotomic extension of $\mathbb{Q}$.

Properties

1. $\mathrm{Gal}(\mathbb{Q}_{cyc}/\mathbb{Q}) \simeq \varprojlim_n \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) \simeq \varprojlim_n \left(\mathbb{Z}/p^{n+1}\right)^* \simeq \mathbb{Z}/(p-1) \times \mathbb{Z}_p \, .$

2. ramified only at ∞ and p, in particular $\mathbb{Q}_{cyc}/\mathbb{Q}_0$ is ramified (totally) only at p.

$\mathbb{Q}_{cyc}^{\mathrm{Gal}(\mathbb{Q}_0/\mathbb{Q})}$ is called the **cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$**.
For any number field K, $K\mathbb{Q}_{cyc}^{\mathrm{Gal}(\mathbb{Q}_0/\mathbb{Q})}$ is the **cyclotomic $\mathbb{Z}_p$-extension of K**.
The **Iwasawa algebra** is the ring

$$\Lambda := \varprojlim_n \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}_0)] = \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}_{cyc}/\mathbb{Q}_0)]] \simeq \mathbb{Z}_p[[T]] \ .$$

The last isomorphism is non-canonical and given by $\gamma \mapsto T-1$ where γ is a chosen topological generator of $\mathrm{Gal}(\mathbb{Q}_{cyc}/\mathbb{Q}_0)$.

Cyclotomic extensions: function fields

- $F := \mathbb{F}_q(\theta)$ with $q = p^r \geqslant 3$ and fix $\frac{1}{\theta}$ as the prime at ∞.
- Let $A := \mathbb{F}_q[\theta]$ and fix a prime $\mathfrak{p}$ of A of degree d.
- Let Φ be the **Carlitz module** associated to A: it is an $\mathbb{F}_q$-linear ring homomorphism

$$\Phi : A \to F\{\tau\}$$

$$\theta \mapsto \Phi_\theta = \theta\tau^0 + \tau$$

$$\theta^2 \mapsto \Phi_{\theta^2} = (\theta\tau^0 + \tau)(\theta\tau^0 + \tau) = \theta^2\tau^0 + \theta\tau + \tau\theta + \tau^2 = \theta^2\tau^0 + (\theta + \theta^q)\tau + \tau^2 \ ,$$

  where $F\{\tau\}$ is the skew polynomial ring with $\tau f = f^q\tau$ for any $f \in F$.
- For any ideal $\mathfrak{a}$ of A write

$$\Phi[\mathfrak{a}] := \{x \in \overline{F} \mid \Phi_a(x) = 0 \ \forall\, a \in \mathfrak{a}\} \ ,$$

  it is an A-module isomorphic to $A/\mathfrak{a}$ and such that $\mathrm{Gal}(F(\Phi[\mathfrak{a}])/F) \simeq (A/\mathfrak{a})^*$.
  Example: $\Phi[(\theta)] \simeq \mathbb{F}_q$, indeed

$$\Phi_\theta(a) = \theta a + a^q = 0 \iff a = 0 \text{ or } a^{q-1} = -\theta \ .$$

  Moreover $F(\Phi[(\theta)]) = F(\sqrt[q-1]{-\theta})$ and $\mu_{q-1} \subset \mathbb{F}_q$, so $\mathrm{Gal}(F(\Phi[(\theta)])/F) \simeq \mathbb{F}_q^*$.

Cyclotomic extensions: function fields

For any $n \in \mathbb{N}$, let

- $F_n := F(\Phi[\mathfrak{p}^{n+1}])$

$$F \subset F_0 \subset F_1 \subset \cdots \subset F_n \subset \cdots \subset \bigcup F(\Phi[\mathfrak{p}^n]) = F(\Phi[\mathfrak{p}^\infty]) \ .$$

- $\mathcal{F} := \bigcup F_n$ the $\mathfrak{p}$-cyclotomic extension of F.

Properties

1. $$\begin{aligned} \mathrm{Gal}(\mathcal{F}/F) &\simeq \varprojlim_n \mathrm{Gal}(F_n/F) \simeq \varprojlim_n \left(A/\mathfrak{p}^{n+1}\right)^* \\ &\simeq \mathrm{Gal}(F_0/F) \times \mathrm{Gal}(\mathcal{F}/F_0) \simeq \mathbb{Z}/(q^d-1) \times \mathbb{Z}_p^\infty := \Delta \times \Gamma \end{aligned} \ .$$
2. ramified only at ∞ and $\mathfrak{p}$, in particular $\mathcal{F}/F_0$ is ramified (totally) only at $\mathfrak{p}$ and the inertia group of ∞ is $\mathbb{F}_q^* \hookrightarrow \Delta$ (note $|\Delta| = q^{\deg(\mathfrak{p})} - 1 = q^d - 1$).

The extension $\mathcal{F}^\Delta/F$ is a $\mathbb{Z}_p^\infty$-extension of F *cyclotomic at* $\mathfrak{p}$.
The **Iwasawa algebra** is the ring

$$\Lambda := \varprojlim_n \mathbb{Z}_p[\mathrm{Gal}(F_n/F_0)] = \mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T_n \ : \ n \in \mathbb{N}]] \ .$$

Iwasawa modules: global fields

F a global field and E/F a finite extension

- $\mathcal{C}\ell^{\,0}(E)$ the p-part of the group of divisor classes of E of degree 0 (class group);
- $L(E)$ the maximal unramified abelian p-extension of E (totally split at ∞);
- $\mathcal{C}\ell^{\,0}(E) \simeq \mathrm{Gal}(L(E)/E)$ via the (canonical) Artin map.

Same notations for infinite extensions $\mathcal{E}/F$ where

$$\Lambda(\mathcal{E}) := \mathbb{Z}_p[[\mathrm{Gal}(\mathcal{E}/F)]] \quad \text{and} \quad \mathcal{C}\ell^{\,0}(\mathcal{E}) := \varprojlim_{E} \mathcal{C}\ell^{\,0}(E)$$

(the limit is on the natural norm maps as E runs among the finite subextensions of $\mathcal{E}/F$).

$$\begin{tikzcd}[column sep=small]
 & L(\mathcal{E}) \arrow[dl, dash, "\mathcal{C}\ell^{\,0}(\mathcal{E})"'] \\
\mathcal{E} \arrow[d, dash, "\mathrm{Gal}(\mathcal{E}/F)"'] & \\
F &
\end{tikzcd}$$

$\mathcal{C}\ell^{\,0}(\mathcal{E}) \lhd \mathrm{Gal}(L(\mathcal{E})/F)$

(a lift of) $\mathrm{Gal}(\mathcal{E}/F)$ acts on $\mathcal{C}\ell^{\,0}(\mathcal{E})$ via conjugation

$\mathcal{C}\ell^{\,0}(\mathcal{E})$ is a $\Lambda(\mathcal{E})-$module

Iwasawa modules: global fields

Theorem (Iwasawa 60's, Greenberg 70's,...)

Let $\mathcal{E}/F$ be a $\mathbb{Z}_p^d$-extension ($d < \infty$), then $\mathcal{C}\ell^0(\mathcal{E})$ is a finitely generated torsion $\Lambda(\mathcal{E})$-module.

A f.g.t. $\Lambda(\mathcal{E})$-module N is *pseudo-null* if $ht(Ann_{\Lambda(\mathcal{E})}(N)) \geqslant 2$ (i.e., N has at least 2 relatively prime annihilators).

Theorem (Structure Theorem for f.g.t. Iwasawa modules)

For any f.g.t. $\Lambda(\mathcal{E})$-module M there is a pseudo-isomorphism (i.e. with pseudo-null kernel and cokernel)

$$M \sim_{\Lambda(\mathcal{E})} \bigoplus_{i=1}^{s} \Lambda(\mathcal{E})/(f_i^{e_i}) \qquad \text{i.e.} \qquad \bigoplus_{i=1}^{s} \Lambda(\mathcal{E})/(f_i^{e_i}) \hookrightarrow M \twoheadrightarrow N$$

where the f_i are irreducible elements of $\Lambda(\mathcal{E}) \simeq \mathbb{Z}_p[[T_1, \ldots, T_d]]$, f_i, e_i and s are uniquely determined by M and N is pseudo-null.

Iwasawa modules: characteristic ideals

Definition (Characteristic ideal)

For a f.g.t module as above we define the characteristic ideal *as*

$$Ch_{\Lambda(\mathcal{E})}(M) := \left(\prod_{i=1}^{s} f_i^{e_i} \right) .$$

A f.g.t. $\Lambda(\mathcal{E})$-module N is pseudo-null *(i.e. $N \sim_{\Lambda(\mathcal{E})} 0$)* $\iff$ $Ch_{\Lambda(\mathcal{E})}(N) = (1)$.

Theorem (Iwasawa)

Let K_∞/K be a $\mathbb{Z}_p$-extension of a number field K and and let f_{K_∞} be a (polynomial) generator of $Ch_{\Lambda(K_\infty)}(\mathcal{C}\ell^0(K_\infty))$. Let

- μ *such that* $p^\mu \mid f_{K_\infty}$ *and* $p^{\mu+1} \nmid f_{K_\infty}$*;*
- $\lambda = \deg(f_{K_\infty})$.

Then

$$|\mathcal{C}\ell^0(K_n)| = p^{\mu p^n + \lambda n + O(1)} \quad \forall n \gg 0 .$$

ζ-functions and L-functions (sketch)

Cyclotomic extension of $\mathbb{Q}$

Let Δ be the group of Dirichelet characters of the number field $\mathbb{Q}(\boldsymbol{\mu}_p)$, then

$$\zeta_{\mathbb{Q}(\boldsymbol{\mu}_p)}(s) = \sum_{I \neq 0} N_{\mathbb{Q}(\boldsymbol{\mu}_p)/\mathbb{Q}}(I)^{-s} = \prod_{\chi \in \Delta} \left(\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \right) = \prod_{\chi \in \Delta} L(s,\chi) \ .$$

The special value in $s=1$ is related to the class number and other arithmetically relevant information on the field $\mathbb{Q}(\boldsymbol{\mu}_p)$.

- $\Delta \simeq \mathrm{Gal}(\mathbb{Q}(\boldsymbol{\mu}_p)/\mathbb{Q})$ is cyclic generated by the Teichmüller character ω (recall $\omega(a) \equiv a \pmod{(1-\zeta_p)}$).
- For characters ω^i (i even and nonzero) Iwasawa defined a p-adic L-function $L_p(s,\omega^i)$ which interpolates the special values of $L(s,\omega^i)$.
- Iwasawa proved that there exist power series $f(T,\omega^i) \in \mathbb{Z}_p[[T]] \simeq \Lambda(\mathbb{Q}_{cyc})$ such that

$$L_p(s,\omega^i) = f((1+p)^s - 1, \omega^i) \ .$$

Nowadays many generalizations are known: a similar procedure works for any number field and some of its $\mathbb{Z}_p$-extensions (the ones with more relevant arithmetical meaning).

Main Conjecture

$$\begin{tikzcd}
& {|\mathcal{C}\ell^0(K)|} \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\
\mathcal{C}\ell^0(K_\infty) \arrow[d, dash] & & \text{sp. values of } \zeta \text{ and } L-\text{functions} \arrow[d, dash] \\
(f_{K_\infty}) = Ch_\Lambda(\mathcal{C}\ell^0(K_\infty)) & \text{IMC} \arrow[l] \arrow[r] & p-\text{adic } L-\text{function } L_p \in \Lambda
\end{tikzcd}$$

Consequences/related results:

- special values of ζ-functions;
- arithmetic properties of (generalized) Bernoulli numbers;
- vanishing of the μ-invariant for cyclotomic extensions K_{cyc}/K, i.e., the p-adic L-function is nonzero modulo p (Ferrero-Washington theorem when $K/\mathbb{Q}$ is abelian).

Function field setting:

- **Analytic side** analogues of ζ-functions, Bernoulli numbers and so on (mainly due to Carlitz and Goss);
- **Algebraic side** arithmetic $\mathbb{Z}_p$-extension, geometric $\mathbb{Z}_p^d$-extensions, no arithmetic informations on $\mathbb{Z}_p^\infty$-extensions.

Main Conjecture and BSD

Let E be an elliptic curve (defined over $\mathbb{Q}$) and consider the exact sequence

$$E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow Sel(E/\mathbb{Q})_p \twoheadrightarrow Ш(E/\mathbb{Q})[p^\infty]$$

and the L-function $L(E,s)$ associated to E.
BSD Conjecture

$$ord_{s=1} L(E,s) = rk_{\mathbb{Z}}(E(\mathbb{Q})) \; + \; Ш(E/\mathbb{Q}) \text{ is finite } + \text{ much more }.$$

$$\begin{tikzcd}
 & rk_{\mathbb{Z}}(Sel(E/\mathbb{Q})_p) \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\
Sel(E/\mathbb{Q}_{cyc})_p \arrow[d, dash] & & \text{sp. values of } L(E,s) \arrow[d, dash] \\
Ch_\Lambda(Sel(E/\mathbb{Q}_{cyc})_p) & \text{IMC (Mazur)} \arrow[l] \arrow[r] & p-\text{adic } L-\text{function } L_p(E,T) \in \Lambda
\end{tikzcd}$$

Actually this is the most used strategy to produce results on BSD (Kato, Bertolini-Darmon, Skinner-Urban,...).

Fitting ideals: Greither-Popescu *IMRN '12* and *Crelle '13*

Let $\mathcal{F}/F$ be the $\mathfrak{p}$-cyclotomic extension of $F = \mathbb{F}_q(\theta)$ with $\mathrm{Gal}(F_0/F) =: \Delta$, $\mathrm{Gal}(\mathcal{F}/F_0) =: \Gamma$ and (non-noetherian) Iwasawa algebra $\Lambda(\mathcal{F})$. Let $\mathcal{C}\ell^0(\mathcal{F})$ be the Iwasawa module of $\mathcal{F}$.
We do not have a structure theorem for $\Lambda(\mathcal{F})$-modules (hence no characteristic ideal), so we compute Fitting ideals for the class groups of $F_n = F(\Phi[\mathfrak{p}^{n+1}])$ in the group ring $\mathbb{Z}_p[\mathrm{Gal}(F_n/F)]$.

Definition (Fitting ideal)

Let M be a f.g. R-module. The Fitting ideal *of M over R, $\mathrm{Fitt}_R(M)$ is the ideal of R generated by the determinants of all the (minors of the) matrices of relations for a fixed set of generators of M.*

Let $\overline{\mathbb{F}}_q$ be a fixed algebraic closure of $\mathbb{F}_q$ and fix a topological generator γ for $G_{\mathbb{F}_q} := \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ (the *arithmetic Frobenius*).
Denote by

$$T_p(L) := T_p(Jac(\mathcal{X}_L)(\overline{\mathbb{F}}_q)) \ .$$

the *p-adic Tate module* of the $\overline{\mathbb{F}}_q$-rational points of the Jacobian of the curve $\mathcal{X}_L$ associated with the field L.

Lemma

$$\mathcal{C}\ell^0(F_n) \simeq T_p(F_n)/(1-\gamma^{-1})T_p(F_n) =: T_p(F_n)_{G_{\mathbb{F}_q}} \ .$$

Fitting ideals: Greither-Popescu *IMRN '12* and *Crelle '13*

The module $T_p(F_n)$ fits into the exact sequence

$$T_p(F_n) \hookrightarrow T_p(\mathcal{M}_{S,n}) \twoheadrightarrow L_n$$

where $T_p(\mathcal{M}_{S,n})$ is the p-adic realization of a *motive* $\mathcal{M}_S$ (defined via divisors), L_n is the kernel of the degree map $\mathbb{Z}_p[\overline{\mathbb{F}}_q F_n(S)] \longrightarrow \mathbb{Z}_p$ and $\overline{\mathbb{F}}_q F_n(S)$ is the set of places of $\overline{\mathbb{F}}_q F_n$ lying above places in $S := \{\mathfrak{p}, \infty\}$.

Definition (Stickelberger series)

Let $\mathcal{F}_S =$ be the maximal pro-p-extension of F unramified outside S. The Stickelberger series *is defined as*

$$\Theta_S(X) := \prod_{v \notin S} (1 - \mathrm{Fr}_v^{-1} X^{d_v})^{-1} \in \mathbb{Z}[[\mathrm{Gal}(\mathcal{F}_S/F)]][[X]] \ ,$$

where Fr_v is the (lifting of the) Frobenius of v in $\mathrm{Gal}(\mathcal{F}_S/F)$. Moreover, for any n, we put $\Theta_n(X)$ for the projection of $\Theta_S(X)$ in $\mathbb{Z}[\mathrm{Gal}(F_n/F)][[X]]$.

Theorem (Greither-Popescu *IMRN '12* Theorem 4.3)

$$\mathrm{Fitt}_{\mathbb{Z}_p[G_n][[G_{\mathbb{F}_q}]]}(T_p(\mathcal{M}_{S,n})) = (\Theta_n(\gamma^{-1})) \ .$$

Fitting ideals: class groups

Let χ be a character of $\Delta := \mathrm{Gal}(F_0/F)$ with values in the Witt ring $W \simeq \mathbb{Z}_p[\boldsymbol{\mu}_{q^d-1}]$: we say that χ is **even** is $\chi(\mathbb{F}_q^*) = 1$ and **odd** otherwise. Denote the trivial character by χ_0.
From now on we work on χ-parts of modules M, denoted $M(\chi)$.
Computing the Fitting ideal of $T_p(F_n)(\chi)$ and specializing γ^{-1} to 1 one gets

Proposition (Anglés-B-Bars-Longhi (ABBL) '15)

$$\mathrm{Fitt}_{W[\Gamma_n]}(\mathcal{C}\ell^0(F_n)(\chi)) := \begin{cases} (\Theta_n(1,\chi)) & \text{if } \chi \text{ is odd} \\ \left(\frac{\Theta_n(X,\chi)}{1-X}|_{X=1}\right) & \text{if } \chi \neq \chi_0 \text{ is even} \end{cases} .$$

$$\mathrm{Fitt}_{W[\Gamma_n]}(\mathcal{C}\ell^0(F_n)^\vee(\chi_0)) = \frac{\Theta_n(X,\chi_0)}{1-X}|_{X=1}\left(1, \frac{n(\Gamma_n)}{d}\right)$$

where $\Gamma_n := \mathrm{Gal}(F_n/F_0)$ *and* $n(\Gamma_n) := \sum_{g\in\Gamma_n} g$.

Remark

The ramified prime ∞ causes the $1-X$ for even characters.
The ramified prime $\mathfrak{p}$ causes the extra factor for $\chi = \chi_0$.

Main (algebraic) theorem

Let $\mathcal{C}\ell^0(\mathcal{F}) := \varprojlim_n \mathcal{C}\ell^0(F_n)$ (limit on the norm maps).

Studying kernels and cokernels of the natural norm and inclusion maps between $\mathcal{C}\ell^0(F_n)$ and $\mathcal{C}\ell^0(F_{n-1})$ (in particular norms are surjective and their kernels are gived by augmentation ideals), one verifies that the system of Fitting ideals is coherent with espect to norms.

Theorem (ABBL '15)

$$\mathrm{Fitt}_{W[[\Gamma]]}(\mathcal{C}\ell^0(\mathcal{F})(\chi)) = \varprojlim_n \left(\Theta_n^{\#}(1,\chi)\right) := \left(\Theta_\infty^{\#}(1,\chi)\right) \quad \text{for } \chi \neq \chi_0$$

with

$$\Theta_n^{\#}(1,\chi) := \begin{cases} \Theta_n(1,\chi) & \text{if } \chi \text{ is odd} \\ \dfrac{\Theta_n(X,\chi)}{1-X}\Big|_{X=1} & \text{if } \chi \neq \chi_0 \text{ is even} \end{cases} .$$

Goss-Carlitz ζ-function

Let

- A_+ (resp. $A_{+,n}$) denote the set of monic polynomials of A (resp. monic polynomials of degree n);
- $F_\infty = \mathbb{F}_q((\theta^{-1}))$ the completion of F at ∞ and $\overline{F_\infty}$ its algebraic closure;
- $\mathbb{C}_\infty := \widehat{\overline{F_\infty}}$ the completion of $\overline{F_\infty}$ at the prime dividing ∞;
- $\mathbb{S}_\infty := \mathbb{C}_\infty^* \times \mathbb{Z}_p$.

Definition (Goss-Carlitz ζ-function)

The Goss-Carlitz ζ-function is defined as

$$\zeta_A(s) := \sum_{a \in A_+} a^{-s} = \sum_{n \geq 0} \left(\sum_{a \in A_{+,n}} \langle a \rangle_\infty^{-y} \right) x^{-n} \quad , \ s = (x, y) \in \mathbb{S}_\infty \ ,$$

where

$$a^s = x^{\deg(a)} \langle a \rangle_\infty^y \text{ and } \langle a \rangle_\infty := \frac{a}{\theta^{\deg(a)} sgn(a)} = 1 + \text{ pws of } \frac{1}{\theta} \in \mathbb{F}\left[\frac{1}{\theta}\right]$$

and $sgn(a)$ is the leading coefficient of a.

Stickelberger and $\zeta_A(s)$: interpolation

Let $\mathbb{S}_\infty^+ := \{(x,y) \in \mathbb{S}_\infty \,:\, |x| > 1\}$ be our "half-plane" ($\sim\ \mathbb{C}^+ = \{z \in \mathbb{C} \,:\, Re(z) > 1\}$). Then

- since $a^s = x^{\deg(a)}\langle a\rangle_\infty^y$, one has $|a^{-s}| = |x|^{-\deg(a)}$ and $\zeta_A(s)$ converges on $\mathbb{S}_\infty^+$;
- an element $s = (x,y) \in \mathbb{S}_\infty$ defines a principal quasi-character

$$\varphi_s \,:\, \mathbb{I}_F/F^* \simeq \mathbb{Z} \times U_1(\infty) \times \prod_{v \neq \infty} A_v^* \longrightarrow \mathbb{C}_\infty^*$$

$$(n, \alpha_1, \prod \alpha_v) \xrightarrow{\varphi_s} x^{-n}\alpha_1^y \;;$$

- for any $y \in \mathbb{Z}_p$ consider the map

$$\psi_y \,:\, \mathrm{Gal}(\mathcal{F}_S/F) \overset{cls\ field}{\twoheadrightarrow} U_1(\infty) \overset{(id,y)}{\hookrightarrow} \mathbb{S}_\infty \xrightarrow{a \mapsto a^s} \mathbb{C}_\infty^* \,.$$

  For any $v \notin S$ one has $\psi_y(\mathrm{Fr}_v^{-1}) = \langle \pi_v \rangle_\infty^y$ (where π_v is the monic generator of the prime ideal v) and $\psi_y(\Theta_S)(x)$ converges for any $|x| \geqslant 1$.

Theorem (Stickelberger $\leftrightarrow \zeta_A$, ABBL '15)

For any $s = (x,y) \in \mathbb{S}_\infty^+$,

$$\psi_{-y}(\Theta_S)(x^{-1}) = (1 - \pi_{\mathfrak{p}}^{-s})\zeta_A(s) \,.$$

Stickelberger and $\zeta_A(s)$: convergence

Remarks

- The proof of the previous theorem uses the Euler product formula

$$\zeta_A(s) = \prod_{v \neq \infty} (1 - \pi_v^{-s})^{-1} .$$

- By the properties of the Stickelberger element, the previous theorem can be used to ensure the convergence of ζ_A on the border of the disc $|x| \geqslant 1$.
- One can prove that

$$v_\infty \left(\sum_{a \in A_{+,n}} \langle a \rangle_\infty^y \right) \geqslant p^{n-1}$$

for any $y \in \mathbb{Z}_p$ and any $n \geqslant 1$, thus ensuring the convergence of ζ_A on the whole $\mathbb{S}_\infty$.

The last remark is particularly relevant because we are interested in the values of ζ_A at **negative** integers, i.e.,

$$\zeta_A(-j) := \zeta_A(\theta^{-j}, -j) \quad j \in \mathbb{N} - \{0\}$$

(note that $v_\infty(\theta^{-j}) = -j$ and $|\theta^{-j}| = |1/\theta|^j \leqslant 1$).

Special values of $\zeta_A(s)$

For any $j \in \mathbb{Z}$ and $n \in \mathbb{N}$ let

$$S_n(j) := \sum_{a \in A_{+,n}} a^j \in A \quad \text{and} \quad Z(X,j) := \sum_{n \geqslant 0} S_n(j) X^n \in A[X] \ .$$

Then $\zeta_A(-j) := \zeta_A(\theta^{-j}, -j) = Z(1,j)$ and it has **trivial zeroes** at **even** integers, i.e.,

$$\zeta_A(-j) = Z(1,j) = 0 \quad \text{for } j \geqslant 1,\ j \equiv 0 \pmod{q-1} \ .$$

Definition (Bernoulli-Goss numbers)

For any $j \in \mathbb{N}$, the Bernoulli-Goss numbers $\beta(j)$ *are defined as*

$$\beta(j) := \begin{cases} Z(1,j) & \text{if } j = 0 \text{ or } j \not\equiv 0 \pmod{q-1} \\ -\frac{d}{dX} Z(X,j)_{|X=1} & \text{if } j \geq 1 \text{ and } j \equiv 0 \pmod{q-1} \end{cases}$$

Lemma

For all $j \geq 0$, we have $\beta(j) \equiv 1 \pmod{\theta^q - \theta}$. In particular $\beta(j) \neq 0$.

$\mathfrak{p}$-adic L-function

The characters of $\Delta = \mathrm{Gal}(F_0/F)$ are powers of the Teichmüller $\omega_\mathfrak{p}$ (values in $A_\mathfrak{p}$)

$$a = \omega_\mathfrak{p}(a)\langle a\rangle_\mathfrak{p} \quad \text{with} \quad a \equiv \omega_\mathfrak{p}(a) \pmod{\mathfrak{p}} \quad \text{and} \quad \langle a\rangle_\mathfrak{p} \in U_1 \; .$$

Let $\widetilde{\omega}_\mathfrak{p}$ be the composition of $\omega_\mathfrak{p}$ with the cyclotomic character (with values in W^*).

Definition ($\mathfrak{p}$-adic L-function)

For any $0 \leq i \leq q^d - 2$ *and any* $y \in \mathbb{Z}_p$*, the* $\mathfrak{p}$-adic L-function *is defined as*

$$L_\mathfrak{p}(X, y, \omega_\mathfrak{p}^i) := \sum_{n \geq 0} \left(\sum_{a \in A_{+,n}\,,\ \mathfrak{p} \nmid (a)} \omega_\mathfrak{p}^i(a)\langle a\rangle_\mathfrak{p}^y \right) X^n \in A_\mathfrak{p}[[X]] \; .$$

Remarks

- By estimating the $\mathfrak{p}$-adic valuation of the coefficients of X^n one sees that it converges on $\mathbb{S}_\mathfrak{p} := \mathbb{C}_\mathfrak{p}^* \times \mathbb{Z}_p \times \mathbb{Z}/(q^d - 1)$ (which is the analogue of the previous $\mathbb{S}_\infty$).
- Another (almost unexplored) analogy: the space $\mathbb{S}_\mathfrak{p}$ can be interpreted as the set of principal quasi-characters on $\mathbb{I}_F/F^*$ with values in $\mathbb{C}_\mathfrak{p}^*$.

$L_{\mathfrak{p}}$ and ζ_A : interpolation

Proposition ($L_{\mathfrak{p}} \leftrightarrow \zeta_A$, ABBL '15)

Let $0 \leqslant i \leqslant q^d - 2$, $j \equiv i \pmod{q^d - 1}$, $y \in \mathbb{Z}_p$ *and* $\mathfrak{p} = (\pi_{\mathfrak{p}})$. *Then*

(1) $L_{\mathfrak{p}}(X, j, \omega_{\mathfrak{p}}^i) = (1 - \pi_{\mathfrak{p}}^j X^d) Z(X, j) \in A[X]$ and $L_{\mathfrak{p}}(X, y, \omega_{\mathfrak{p}}^i) \equiv Z(X, i) \pmod{\mathfrak{p}}$.

(2o) $L_{\mathfrak{p}}(1, j, \omega_{\mathfrak{p}}^i) = (1 - \pi_{\mathfrak{p}}^j)\beta(j)$, $i \not\equiv 0 \pmod{q - 1}$

(2e) $\frac{d}{dX} L_{\mathfrak{p}}(X, j, \omega_{\mathfrak{p}}^i)_{|X=1} = -(1 - \pi_{\mathfrak{p}}^j)\beta(j)$, $j \geq 1$, $i \equiv 0 \pmod{q - 1}$.

The proof is easier if one consider the Euler product formula of the two functions. The one for $L_{\mathfrak{p}}$ is

$$L_{\mathfrak{p}}(X, y, \omega_{\mathfrak{p}}^i) = \prod_{v \notin S} (1 - \omega_{\mathfrak{p}}^i(\pi_v) \langle \pi_v \rangle_{\mathfrak{p}}^y X^{\deg(v)})^{-1} .$$

Stickelberger and $L_{\mathfrak{p}}$: Iwasawa Main Conjecture

For any $s=(1,y,i)\in\mathbb{S}_{\mathfrak{p}}$, consider the map

$$\xi_s\ :\ \mathrm{Gal}(\mathcal{F}_S/F)\xrightarrow{res}\mathrm{Gal}(\mathcal{F}/F)\xrightarrow{\kappa}A_{\mathfrak{p}}^*\hookrightarrow\mathbb{S}_{\mathfrak{p}}\xrightarrow{a\mapsto a^s}\mathbb{C}_{\mathfrak{p}}^*\ .$$

Theorem (Stickelberger $\leftrightarrow L_{\mathfrak{p}}$, ABBL '15)

For any $s=(1,y,i)\in\mathbb{S}_{\mathfrak{p}}$ one has

$$\xi_{-s}(\Theta_S)(X)=L_{\mathfrak{p}}(X,y,\omega_{\mathfrak{p}}^i)\ .$$

Note that the left hand side lives in characteristic 0 and the right hand side in characteristic p. This is more evident in the following equivalent statement.

Theorem (Stickelberger $\leftrightarrow L_{\mathfrak{p}}$, Sinnott '08 + ABBL '15)

Let $0\leq i\leq q^d-2$; for any $y\in\mathbb{Z}_p$, the homomorphism $\gamma\mapsto\kappa(\gamma)^y$ induces a map $s_X:(W[[\Gamma]]/p)[[X]]\to C^0(\mathbb{Z}_p,A_{\mathfrak{p}})$ such that

$$s_X(\Theta_\infty(X,\widetilde{\omega}_{\mathfrak{p}}^{-i}))(y)=L_{\mathfrak{p}}(X,-y,\omega_{\mathfrak{p}}^i)\ .$$

Remark

Specializing at $X=1$ the left hand side provides the Fitting ideal of the Iwasawa module $\mathcal{C}\ell^0(\mathcal{F})$, the right hand side interpolates (via $Z(X,i)$) the values of Goss ζ-function.

Ferrero-Washington theorem

Theorem ($\mu = 0$, ABBL '15)

For any $1 \leq i \leq q^d - 2$ *(i.e.* $\omega_{\mathfrak{p}}^i \neq \chi_0$*), one has* $\Theta_\infty^{\#}(1, \widetilde{\omega}_{\mathfrak{p}}^i) \not\equiv 0 \pmod{p}$.

Proof.

- $\beta(j) \neq 0$ and interpolation property (2);
- $L_{\mathfrak{p}}(1, j, \omega_{\mathfrak{p}}^i) \neq 0$ for $j \not\equiv 0 \pmod{q-1}$;
- $\dfrac{d}{dX} L_{\mathfrak{p}}(X, j, \omega_{\mathfrak{p}}^i)_{|X=1} \neq 0$ for $j \geq 1$, $j \equiv 0 \pmod{q-1}$;
- $s_X(\Theta_\infty^{\#}(1, \widetilde{\omega}_{\mathfrak{p}}^i))(-j) \neq 0$ and s injective on $W[[\Gamma]]/pW[[\Gamma]]$.

$\square$

Since $\Theta_\infty^{\#}(1, \widetilde{\omega}_{\mathfrak{p}}^i)$ is the limit of the $\Theta_n^{\#}(1, \widetilde{\omega}_{\mathfrak{p}}^i)$ we can define

$$N_{\mathfrak{p}}(i) := \mathrm{Inf}\left\{ n \geqslant 0 \, : \, \Theta_n^{\#}(1, \widetilde{\omega}_{\mathfrak{p}}^i) \neq 0 \pmod{p} \right\} .$$

Bernoulli-Goss numbers

Theorem (Arithmetic properties of B-G numbers, ABBL '15)

Let $1 \leq i \leq q^d - 2$ *, then*

$$N_{\mathfrak{p}}(i) \leqslant \mathrm{Inf}\{ v_{\mathfrak{p}}(\beta(j)) \,:\, j \equiv -i \pmod{q^d - 1} \} \,.$$

Proof.

- Let $1 \leq i \leq q^d - 2$ and put

$$m_{\mathfrak{p}}(i) := \begin{cases} \mathrm{Inf}\left\{ v_{\mathfrak{p}}(L_{\mathfrak{p}}(1, y, \omega_{\mathfrak{p}}^i)) \,:\, y \in \mathbb{Z}_p \right\} & \text{for } i \not\equiv 0 \pmod{q-1} \\ \mathrm{Inf}\left\{ v_{\mathfrak{p}}\left(\frac{d}{dX} L_{\mathfrak{p}}(X, y, \omega_{\mathfrak{p}}^i)_{|X=1} \right) \,:\, y \in \mathbb{Z}_p \right\} & \text{for } i \equiv 0 \pmod{q-1} \end{cases} .$$

- Interpolation property (2) $\Longrightarrow m_{\mathfrak{p}}(i) = \mathrm{Inf}\left\{ v_{\mathfrak{p}}(\beta(j)) \,:\, j \geq 1,\ j \equiv i \pmod{q^d - 1} \right\}$.
- Using the link with MC

$$N_{\mathfrak{p}}(i) \leqslant m_{\mathfrak{p}}(-i) = \mathrm{Inf}\{ v_{\mathfrak{p}}(\beta(j)) \,:\, j \equiv -i \pmod{q^d - 1} \} \,.$$

$\square$

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